DEVELOPMENT OF A MATHEMATICAL MODEL FOR THE CLASS V FLEXTENSIONAL UNDERWATER ACOUSTIC TRANSDUCER SHELL

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ABSTRACT

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INTRODUCTION

The flextensional underwater acoustic transducer concept is presently undergoing extensive analytical evaluations. In general, the various flextensional designs can be placed in one of five different classes [1]. This report describes an analytical model developed for the Class V flextensional underwater acoustic transducershell. A picture of this concept is shown in Figure 1 and a detailed sketch of a typical design is given in Figure 2.

Flextensional transducer designs of the general type shown in Figure 1 was originally proposed as having possible applications as a sonobuoy transducer. As shown in Figure 2 this type of design consists of two shallow spherical shells bonded at a boundary and a thin piezoelectric disk joined at this boundary by utilizing an epoxy cement. The piezoelectric disk is isolated electrically from the two shells by removing the silver electrodes beyond the region of contact between the shells and the ceramic disk. Sufficient epoxy is applied so as to firmly attach the disk to the inside shell boundary. Two small holes are drilled through the shells and serve as entrance for the electrical leads to the electrodes plated onto the ceramic disk.

Although this class of flextensional designs is designed primarily to be used as a sonobuoy transducer there are special environments where this type of transducer could be used mainly as a source of acoustic energy. In this instance an additional clamping load would have to be applied around the boundary.

If the Class V type of flextensional design is to function satisfactory as a sensor then the flat portion of the response curve needs to be as broad as possible. Associated with most attempts to increase

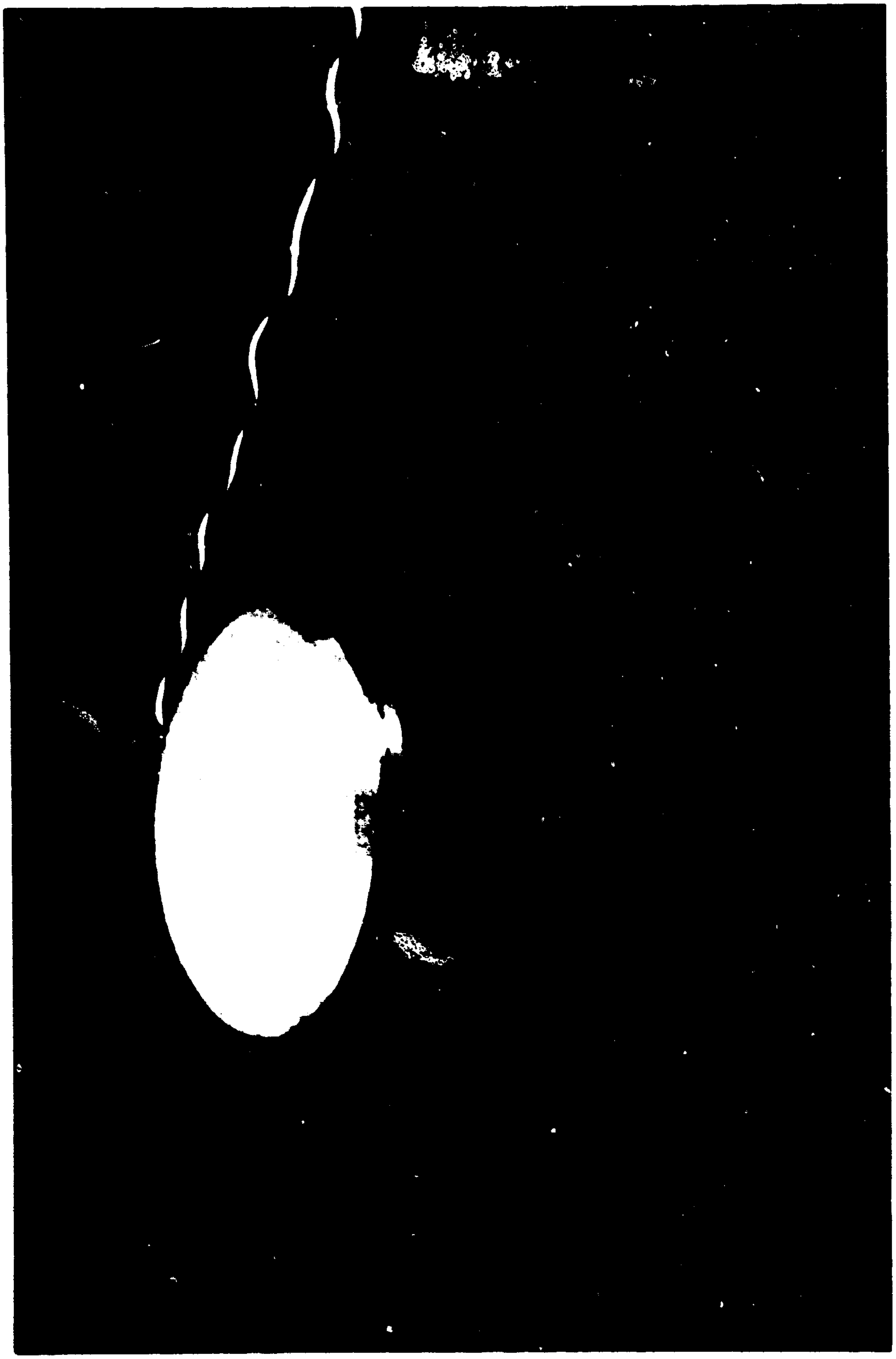

FIG. 1. PICTURE OF FLEXTENSIONAL SHALLOW SHELL SONOBUOY

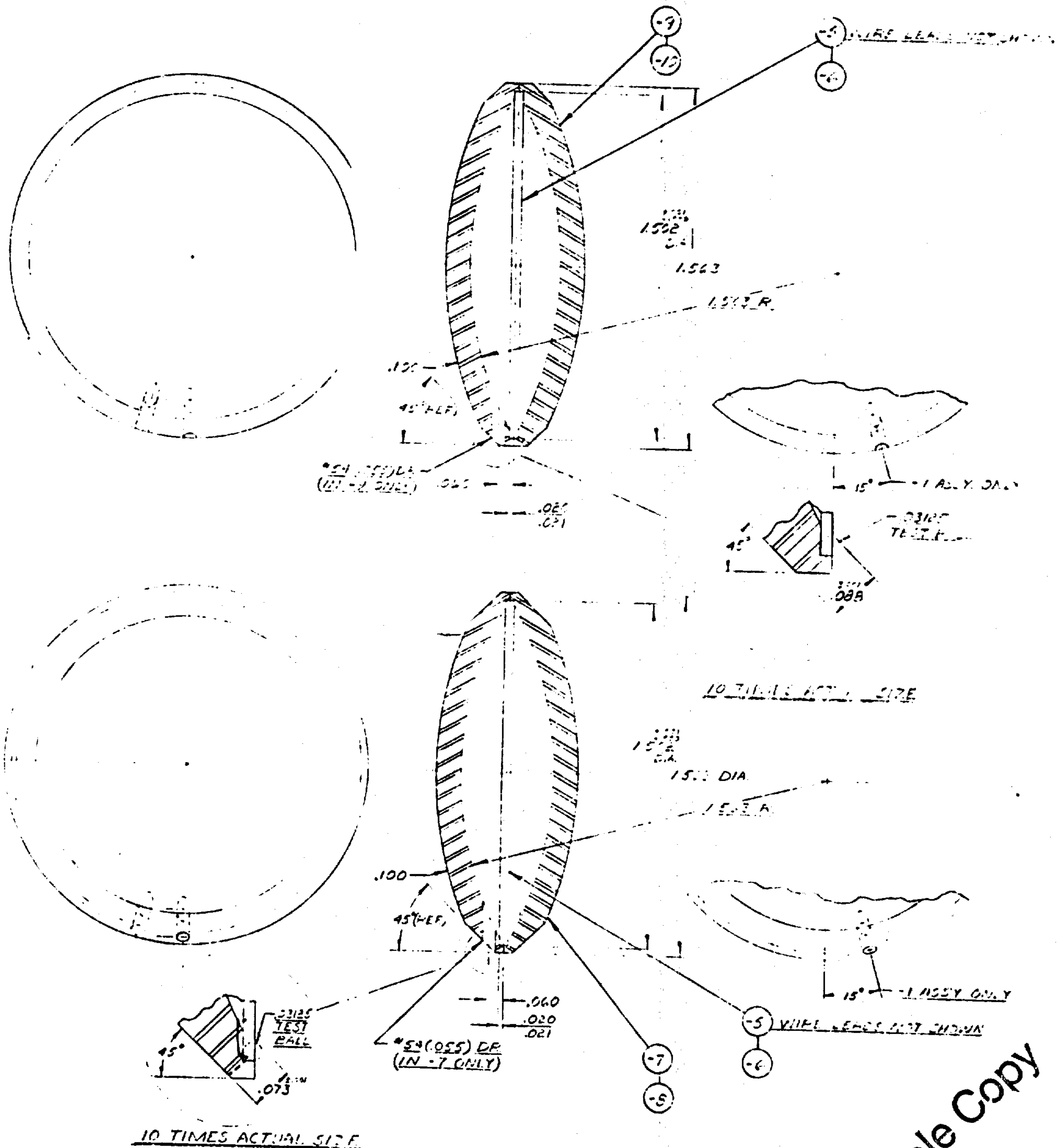

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FIG. 2. TYPICAL GEOMETRY

the sensitivity of the shallow shell concept is a reduction in the systems fundamental resonant frequency. Of course, a reduction in the fundamental resonant frequency reduces the usable frequency range of the concept. An empirical equation has been derived that effectively predicts the sensitivity of this concept below the fundamental resonant frequency.

If the total performance capacity for this type of transducer design is to be fully realized then it is necessary that a detailed mathematical model be developed. The purpose for this report is to present an analytical model that can predict the dynamic characteristics for this type of sonobuoy shell design. Of course once a dynamic model of the shell exists, then conbining such a model with the solution developed for a thin piezoelectric disk with an arbitrary impendance on the boundary [3] will result in a math model for the complete system in air. If in addition the external acoustic loads are determined by utilizing a numerical technique such as has been developed by Hess [4], then a complete math model will exist.

The math model described by the main body of this report assumes that the edges of the shells are horizontally guided-pinned. In attempting to determine an empirical equation that would consistently predict the receiving sensitivity of the shallow shell concept considerable difficulty was encountered. One reason for this difficulty was the inability to establish the degree of clamping between the surrounding shells and the ceramic disk. Possible causes of this variation are the size of the shell-ceramic contact area, the variation in the stiffness of the bond joint which holds the ceramic and the two shells together and the

variation in the thermal expansions between the shells and piezoelectric disk during the curing stage. It has been initially assumed that the bond joint acts more as a pinned boundary than as a clamped one. Also from a practical standpoint it has been necessary to taper the edges of the shell as is shown in Figure 2.

STRAIN ENERGY IN A SPHERE

Static Potential Energy of a Sphere

Using the results obtained by Langhaar [5], one can specialize them to obtain the strain energy of a sphere or spherical section. Noting Fig. 3 which is the same as McDonald's [6], one can write the parametric eqs. of a sphere. These are

$$z = a \cos\theta,$$
$$x = a \sin\theta \cos Q, \qquad (1)$$
and $$y = a \sin\theta \sin Q.$$

In eqs. (1), the parameters are choosen

$$x = \theta \quad \text{and} \quad y = Q, \qquad (2)$$

and the radius of the sphere has been denoted as "a". Using Langhaar's approach along with (2) above, one finds

$$E = X_x^2 + y_x^2 + z_x^2 = X_\theta^2 + y_\theta^2 + z_\theta^2,$$

and from (1)

$$X_\theta = a \cos\theta \cos Q,$$
$$y_\theta = a \cos\theta \sin Q, \qquad (3)$$
$$z_\theta = -a \sin\theta,$$

so that

$$E = a^2$$

Likewise, one finds

$$G = X_Q^2 + y_Q^2 + z_Q^2$$

where

$$X_Q = -a \sin\theta \sin Q,$$
$$y_Q = a \sin\theta \cos Q, \qquad (4)$$
$$z_Q = 0 \quad ,$$

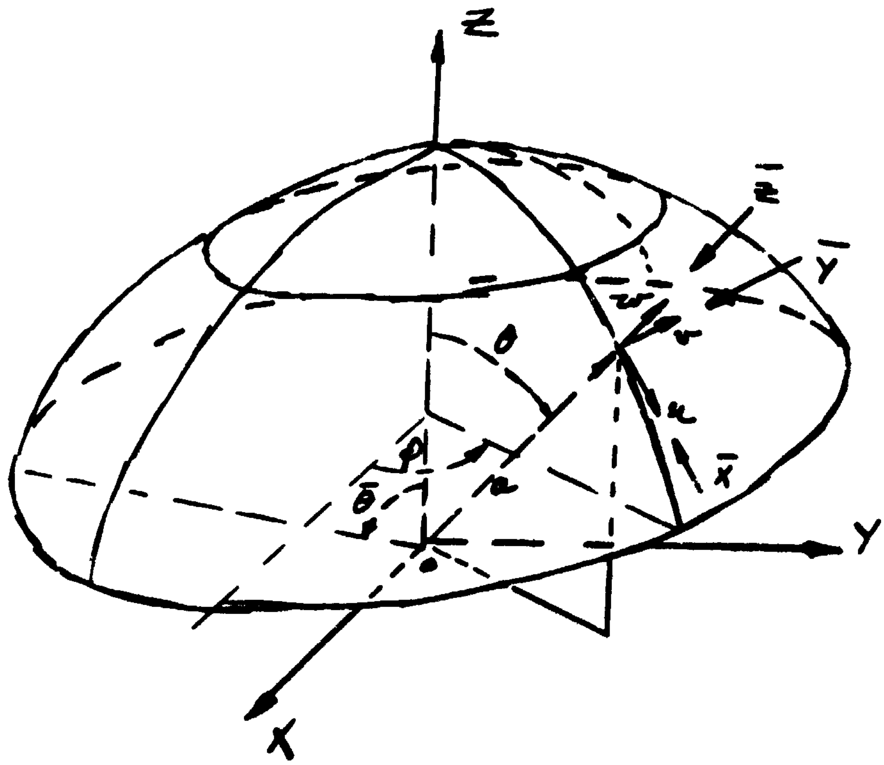

Figure 3

Coordinate System

so that

$$G = a^2 \sin^2 \theta \tag{5}$$

Now, neglecting the quadratic terms, and the terms $(\varepsilon - 2a^2/E)\times$ w and $(\vartheta - 2g^2/G)\,w$ one obtains

$$\left.\begin{aligned}
A_1 &= \sqrt{E}\,u_\theta + \frac{E_\phi v}{2\sqrt{G}} - e w, \\
B_1 &= \sqrt{G}\,v_\phi + \frac{G_\theta u}{2\sqrt{E}} - g w, \\
C_1 &= \frac{1}{2}\left[\sqrt{E}\,u_\phi + \sqrt{G}\,v_\theta - \frac{E_\phi u}{2\sqrt{E}} - \frac{G_\theta v}{2\sqrt{G}}\right]
\end{aligned}\right\} \tag{6}$$

and

$$\left.\begin{aligned}
A_2 &= \left(\frac{eE_\theta}{E} - e_\theta\right)\frac{u}{\sqrt{E}} + \left(\frac{eE_\phi}{E} - e_\phi\right)\frac{v}{\sqrt{G}} + \frac{E_\theta w_\theta}{2E} \\
&\quad - \frac{E_\phi w_\phi}{2G} - w_{\theta\theta}, \\
B_2 &= \left(\frac{gG_\theta}{G} - g_\theta\right)\frac{u}{\sqrt{E}} + \left(\frac{gG_\phi}{G} - g_\phi\right)\frac{v}{\sqrt{G}} - \frac{G_\theta w_\theta}{2E} \\
&\quad + \frac{G_\phi w_\phi}{2G} - w_{\phi\phi},
\end{aligned}\right\} \tag{7}$$

$$\left.\begin{aligned}
C_2 &= \left(\frac{e}{E} - \frac{g}{G}\right)\sqrt{G}\,v_\theta - \left(\frac{e}{E} - \frac{g}{G}\right)\frac{G_\theta v}{2\sqrt{G}} + \frac{E_\phi w_\theta}{2E} \\
&\quad + \frac{G_\theta w_\phi}{2G} - w_{\theta\phi},
\end{aligned}\right\} \tag{7}$$

$$C'_{2} = -\left(\frac{e}{E} - \frac{g}{G}\right)\sqrt{E}\, u_{Q} + \left(\frac{e}{E} - \frac{g}{G}\right)\frac{E_{\theta} v}{2\sqrt{G}} + \frac{E_{Q} w_{\theta}}{2E}$$

$$+ \frac{G_{\theta} w_{Q}}{2G} - w_{\theta Q}\,.$$

By looking at (6) and (7) it is seen that e & g are needed so that specializing his ellipsoid of revolution's results (a=b), one obtains

$$\left.\begin{aligned} \alpha &= a E^{-1/2} \sin\theta \cos Q,\ \beta = a E^{-1/2} \cos\theta \sin Q, \\ \gamma &= a E^{-1/2} \cos\theta, \end{aligned}\right\} \quad (8)$$

hence

$$\left.\begin{aligned} e &= -a^{2} E^{-1/2}, \\ g &= -a^{2} E^{-1/2} \sin^{2}\theta. \end{aligned}\right\} \quad (9)$$

On using (3) and (5) equations (8) and (9) become

$$\left.\begin{aligned} \alpha &= \sin\theta \cos Q,\ \beta = \sin\theta \sin Q, \\ \gamma &= \cos\theta, \end{aligned}\right\} \quad (10)$$

and

$$\left.\begin{aligned} e &= -a \\ g &= -a \sin^{2}\theta. \end{aligned}\right\} \quad (11)$$

Hence using (3), (5), and (11) in (6) gives

$$A_1 = a(u_\theta + w), \tag{12a}$$

$$B_1 = a(v_\phi + u\cos\theta + w\sin\theta)\sin\theta,$$

$$C_1 = \frac{a}{2}(u_\phi + v_\theta \sin\theta - v\cos\theta).$$

Likewise (7) gives

$$A_2 = -w_{\theta\theta}, \tag{12b}$$

$$B_2 = -w_\theta \sin\theta\cos\theta - w_{\phi\phi},$$

$$C_2 = \frac{\cos\theta}{\sin\theta} w_\phi - w_{\theta\phi} = w_\phi \cot\theta - w_{\theta\phi},$$

$$C_2' = w_\phi \cot\theta - w_{\theta\phi}.$$

Eqs. (12a) and (12b) agree with (21) and (22) obtained by Langhaar [5] on pg. 187.

Therefore the energy due to stretching for a sphere is as follows:

$$V_1 = \frac{\mu}{1-\nu}\int_0^{2\pi}\int_0^{\bar{\theta}} \Big\{(u_\theta + w)^2 + \csc^2\theta(v_\phi + u\cos\theta + w\sin\theta)^2 + 2\nu\csc\theta(u_\theta + w)(v_\phi + u\cos\theta + w\sin\theta) + \frac{1}{2}(1-\nu)\csc^2\theta(u_\phi + v_\theta\sin\theta - v\cos\theta)^2\Big\} h \sin\theta \, d\theta \, d\phi. \tag{13}$$

It should be noted that (13) agrees with eq. (2.1.1) of McDonald's [6] paper. (McDonald uses $\bar{u}$, $\bar{v}$, and $\bar{w}$ meaning u, v, w).

The bending energy, V_2, can be similarily obtained,

$$v_2 = \frac{\mu}{12(1-\nu)} \int_0^{2\pi}\int_0^{\bar{\theta}} \Big\{ w_{\theta\theta}^2 + \csc^4\theta (w_\theta \sin\theta\cos\theta + w_{QQ})^2 + 2\nu \csc^2\theta (w_{\theta\theta})(w_\theta \sin\theta\cos\theta + w_{QQ}) + 2(1-\nu)\csc^2\theta (w_Q \cot\theta - w_{\theta Q})^2 \Big\} \frac{h^3}{a^2} \sin\theta \, d\theta \, dQ . \tag{14}$$

Eqs. (14) is also seen to agree with McDonald's [6] eqs. (2.1.2).

The potential energy of the external forces, Ω, denoted in Fig. 1 by $\bar{X}, \bar{Y}, \bar{Z}$ readily give

$$\Omega = -a^2 \int_0^{2\pi}\int_0^{\bar{\theta}} (\bar{X} u + \bar{Y} v + \bar{Z} w) \sin\theta \, d\theta \, dQ . \tag{15}$$

Hence, using (13), (14) and (15), the total potential energy V of the system is

$$V = v_1 + v_2 + \Omega . \tag{16}$$

Exact Solution In Circumferential Direction

The following middle surface displacement fields are assumed:

$$u(\theta, Q) = \sum_{m=0,1,2,\dots}^{\infty} u_m(\theta) \cos mQ ,$$
$$v(\theta, Q) = \sum_{m=0,1,2,\dots} v_m(\theta) \sin mQ , \tag{17}$$

and

$$w(\theta, Q) = \sum_{m=0,1,2,\dots} w_m(\theta) \cos mQ .$$

These expressions can then be substituted into equations (13) and (14).
To use (17) in (13) and (14), one should note the derviatives with respect to θ,ϕ of (17). To differentiate the series in (17) one assumes all the necessary conditions as noted in Widder's [7] Advanced Calculus, pg. 305, extended to two variables.

Hence,

$$\left.\begin{aligned}
u_\theta(\theta,\phi) &= \sum_{m=0,1,\ldots}^{\infty} u'_m(\theta)\cos m\phi\,, \\
u_\phi(\theta,\phi) &= -\sum_{m=0,1,\ldots}^{\infty} m\,u_m(\theta)\sin m\phi\,, \\
v_\theta(\theta,\phi) &= \sum v'_m(\theta)\sin m\phi\,, \\
v_\phi(\theta,\phi) &= \sum m\,v_m(\theta)\cos m\phi\,, \\
w_\theta(\theta,\phi) &= \sum w'_m(\theta)\cos m\phi\,,
\end{aligned}\right\} \quad (18)$$

and

$$w_\phi(\theta,\phi) = -\sum m\,w_m(\theta)\sin m\phi\,,$$

where

$$u'_m(\theta) = \frac{du_m(\theta)}{d\theta}\,, \text{ etc.}$$

Consider the first term of (13) and using (17) and (18),

$$(u_\theta + w)^2 = \left[\sum_m (u'_m + w_m)\cos m\phi\right]^2.$$

Taking the integral w. r. t. ϕ inside the Σ sign and utilizing orthogonality gives

$$\int_0^{2\pi} (u_\theta + w)^2 \, d\Omega = \sum_{m=0,1,2,\dots}^{\infty} (u'_m + w_m^2)\pi . \tag{19}$$

Similarily one can work with the remaining term of equation (13).

Therefore the strain energy due to membrane becomes

$$\begin{aligned} V_1 = \frac{\mu\pi}{1-\nu} \int_0^{\bar{\theta}} \sum_{m=0,1,2,\dots}^{\infty} \Big[& (u'_m + w_m)^2 + \csc^2\theta (m v_m \\ & + u_m \cos\theta + w_m \sin\theta)^2 + 2\nu \csc\theta (u'_m \\ & + w_m)(m v_m + u_m \cos\theta + w_m \sin\theta) + \\ & + \frac{1}{2}(1-\nu)\csc^2\theta(-m u_m - v'_m \sin\theta + v_m \cos\theta)^2 \Big] \\ & h \sin\theta \, d\theta . \end{aligned} \tag{20}$$

Similarly the strain energy due to bending is

$$\begin{aligned} V_2 = \frac{\mu\pi}{12(1-\nu)} \int_0^{\bar{\theta}} \sum_{m=0,1,2,\dots}^{\infty} \Big[& (w''_m)^2 + \csc^4\theta (w'_m \sin\theta \cos\theta \\ & - m^2 w_m)^2 + 2\nu^2 \csc^2\theta (w''_m)(w'_m \sin\theta\cos\theta \\ & - m^2 w_m) + 2(1-\nu)\csc^2\theta (m w_m \cot\theta - m w' \quad)^2 \Big] \\ & \frac{h^3}{a^2} \sin\theta \, d\theta . \end{aligned} \tag{21}$$

Consider the following substitutions for the load terms:

$$\left.\begin{aligned} \bar{X} &= X(\theta) \sum_{m=0,1,2,\dots}^{\infty} \beta_m \cos m\Omega , \\ \bar{Y} &= Y(\theta) \sum_{m=0,1,2,\dots}^{\infty} \beta_m \sin m\Omega , \\ \bar{Z} &= Z(\theta) \sum_{m=0,1,2,\dots}^{\infty} \beta_m \cos m\Omega , \end{aligned}\right\} \tag{22}$$

where β_m = Fourier coefficient depending on the circumferential distribution of the load. Using (17) and (23) gives using orthogonality

$$\Omega = -\pi a^2 \int_0^{\bar{\theta}} \sum_{m=0,1,2,\dots}^{\infty} \beta_m (\bar{x} u_m + \bar{y} v_m + \bar{z} w_m) \sin\theta \, d\theta \qquad (23)$$

Physical Interpretation of Assumed Displacements and Loads

Consider the case m=0 of eqs. ( ) and (23). For (17)

$$u(\theta, \phi) = u(\theta),$$
$$v(\theta, \phi) = 0,$$
$$\text{and } w(\theta, \phi) = w(\theta). \qquad (24)$$

From (24) the displacements are independent of ϕ and are thus rotionally symmetrical about the axis of revolution. In this case the displacement of any point is the same as every other point on the same latitude. Thus one can show a w-displacement on a cross-section of the spherical cap to be something similar to Fig. which could be rotated about z to give the shell configuration.

Likewise, the load, (23) reduces to

$$\bar{x} = \beta_c x(\theta),$$
$$\bar{y} = 0 \qquad (25)$$
$$\text{and } \bar{z} = \beta_c z(\theta),$$

where $x(\theta)$ to $z(\theta)$ are forces per unit area. Hence from (17) in the case of rotational symmetry, there is zero load in the y-direction. And in a cross-section view which again could be rotated about z, one notes that the loading is similar to that shown in Fig. 5 .

One can next observe the m=1 term and its effect on displacements. Eq. (17) gives

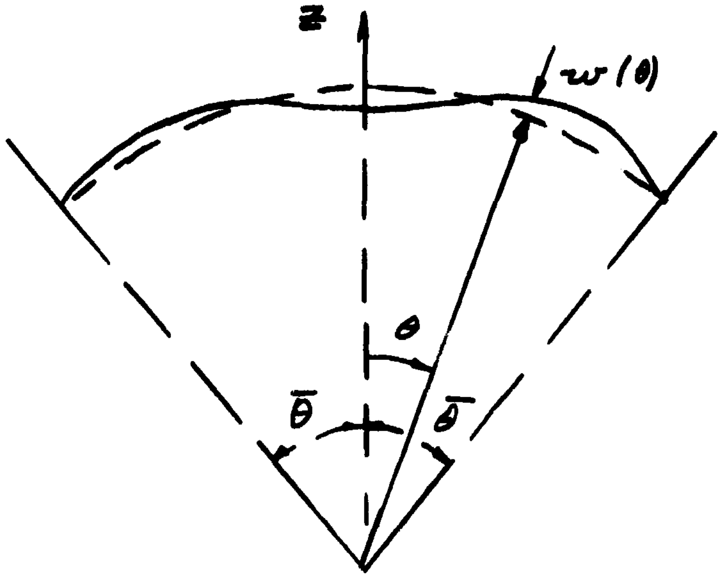

Figure 4 Typical w-displacement for cross-section undergoing rotationally symmetrical vibrations

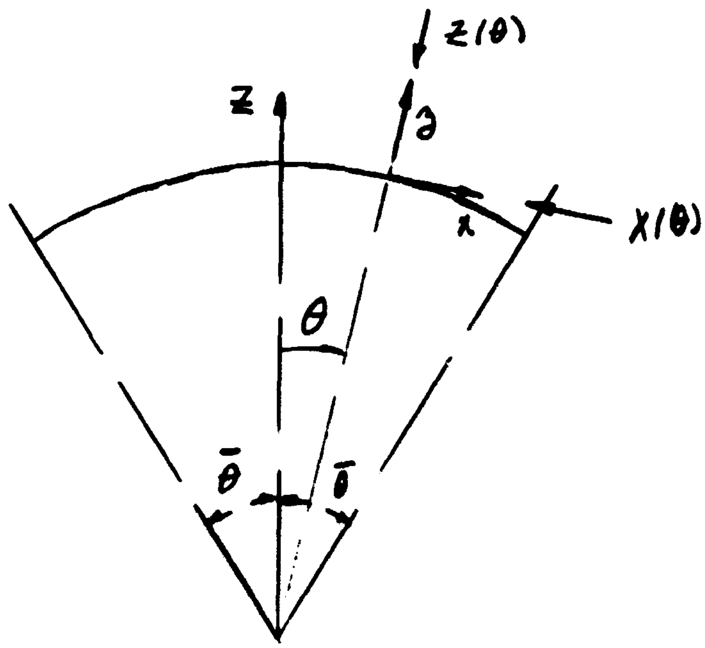

Figure 5 Cross-section view of external loads for rotationally symmetric vibrations

$$
\begin{aligned}
u(\theta,\phi) &= u_1(\theta)\cos\phi, \\
v(\theta,\phi) &= v_1(\theta)\sin\phi, \qquad (26) \\
w(\theta,\phi) &= w_1(\theta)\cos\phi.
\end{aligned}
$$

Thus if one assumes some θ_1, where $0<\theta,<\bar{\theta}$, to be the latitude to be observed, and observing only the v-displacement, then the v-displacement as a function of ϕ on the $\theta=\theta_1$ latitude is as shown in Fig.6.
It should also be noted that eqs.(26) relates the stretching type of motion in some manner. Thus when u and w are taking on their maximum and minimum values, the v-displacements (circumferential) is zero; however, v is maximum when u and w are zero so as to allow u and w to take on their maximum and minimum values by stretching at this point.

One can likewise carry on this analysis and look at other values of m, however, it is felt that a good physical grasp should have already been obtained.

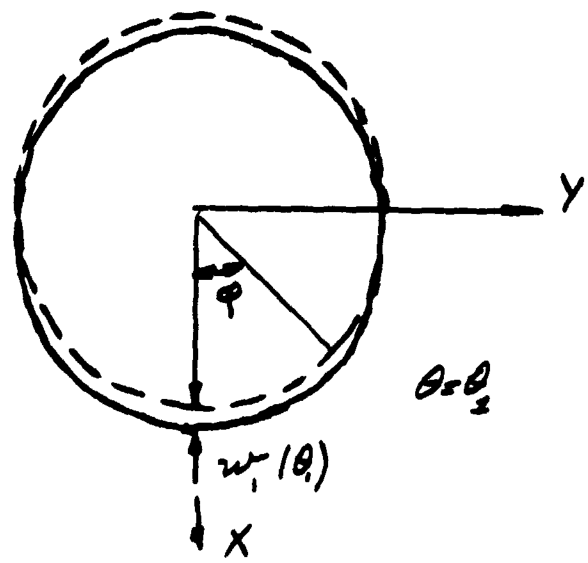

Figure 6

Typical w-displacement for m=1 and some $\theta=\theta_1$

APPROXIMATIONS FOR THE DERIVATIVES IN THE BOUNDARY SEGMENTS

The crown point of the dome, the point of zero meridian (co-latitude), is considered a boundary. Hence a statement must be made about the displacement at this node.

Two cases are distinguished. The first is that in which the vibrations are rotationally symmetrical about the axis of revolution. In this case the displacement of any point is the same as every other point on the same meridian. This case arises when $m = 0$. Therefore, the only possible motion at the crown is one in which the crown node is displaced in a radial direction only, with no accompanying tangential movement, and the slope remains zero. This is expressed mathematically as

$$u(0) = v(0) = 0 \quad ,$$

and

$$\left. \frac{\partial w}{\partial \theta} \right|_{\theta = 0} = 0 \quad . \tag{27}$$

McDonald [6] chooses to use the finite difference expression for the first derivative having an order of error of h^2 while the second derivative expression is the one with an order of error of h. Since he starts at $\theta = 0$ and proceeds positive along the segments then the forward difference is thus the one used. Hence for the general displacement, the first and second derivatives are

$$q_0' = \frac{1}{4}\alpha(-3q_0 + 4q_1 - q_2), \tag{28}$$

and

$$q_0'' = \frac{1}{4}\alpha^2(q_0 - 2q_1 + q_2), \tag{29}$$

where $h = 2\Delta\theta = 2/\alpha$. Hence using the B. C. given by (27) in connection with (28) and (29),

$$u_0 = v_0 = 0, \tag{30}$$

so that

$$\begin{aligned} u_0' &= \frac{1}{4}\alpha(4u_1 - u_2), \\ u_0'' &= \frac{1}{4}\alpha^2(-2u_1 + u_2), \\ v_0' &= \frac{1}{4}\alpha(4v_1 - v_2), \\ v_0'' &= \frac{1}{4}\alpha^2(-2v_1 + v_2). \end{aligned} \tag{31}$$

Also since $w_0' = 0$,

$$w_2 = 4w_1 - 3w_0, \tag{32}$$

so that

$$w_0'' = \frac{1}{4}\alpha^2(w_0 - 2w_1 + w_2).$$

Now using (32) $w_0'' = \frac{1}{2}\alpha^2(-w_0 + w_1)$. (33)

In the second case, since the displacements are in general a function of ϕ, the longitudinal angle, then they must be zero at the crown or the displacement at the crown would be different for each value of ϕ. Hence

$$u(0) = v(0) = w(0) = 0, \tag{34}$$

so that eqs. (2) and (3) become

$$\left.\begin{aligned} q_0' &= \frac{1}{4}\alpha(4q_1 - q_2), \\ \text{and} \quad q_0'' &= \frac{1}{4}\alpha^2(-2q_1 + q_2). \end{aligned}\right\} \tag{35}$$

The boundary conditions at the lower edge, $\bar{\theta}$, depend on the type of support. In the case of the clamped-edge dome studied by McDonald [6], the 3 displacement components and the derivative of the radial displacement must vanish, i.e.

$$\left.\begin{aligned} u(\bar{\theta}) = v(\bar{\theta}) &= w(\bar{\theta}) = 0, \\ \text{and} \quad \left.\frac{\partial w}{\partial \bar{\theta}}\right|_{\theta=\bar{\theta}} &= 0. \end{aligned}\right\} \tag{36}$$

Using the same order of errors as in the crown boundary condition, the backward finite difference expressions are

$$q'_N = \frac{1}{4}\alpha(3q_N - 4q_{N-1} + q_{N-2}),$$

and

$$\left.\begin{aligned} \\ q''_N = \frac{1}{4}\alpha^2(q_N - 2q_{N-1} + q_{N-2}). \end{aligned}\right\} \tag{37}$$

Hence

$$\left.\begin{aligned} u'(\bar{\theta}) &= \frac{1}{4}\alpha(u_{N-2} - 4u_{N-1}), \\ u''(\bar{\theta}) &= \frac{1}{4}\alpha^2(u_{N-2} - 2u_{N-1}), \\ v'(\bar{\theta}) &= \frac{1}{4}\alpha(v_{N-2} - 4v_{N-1}), \\ v''(\bar{\theta}) &= \frac{1}{4}\alpha^2(v_{N-2} - 2v_{N-1}). \end{aligned}\right\} \tag{38}$$

Also

$$w'(\bar{\theta}) = \frac{1}{4}\alpha(-4w_{N-1} + w_{N-2}) = 0,$$

so that

$$w''(\bar{\theta}) = \frac{1}{2}\alpha^2 w_{N-1}. \tag{39}$$

One possible boundary condition for our problem is that of a radial guided boundary with the pinned condition(there might be some torsional spring like effect but for now it is neglected). This arises from the physical fact that two of the spherical caps are placed back-to-back so that it is assumed that motion in one is the same as motion in the other. The B. C. is shown more clearly in Figure 7 which denotes a cross-section. Thus from the physical problem, the case of rotational symmetry is the only one of interest, hence it is assumed that m = 0. In this case the boundary conditions on the lower edge are

$$\text{moment at } (\theta = \bar{\theta}) = m_\theta \Big|_{\theta = \bar{\theta}} = 0, \tag{40}$$

and

$$\text{displacement in z-direction} \Big|_{\theta = \bar{\theta}} = 0 \, . \tag{41}$$

There is no boundary condition on v because $v(\theta, \phi) = 0$.

The B. C. given by (40) will be discussed later on; when expressions for moments are written; however, the B. C. given by (41) can be considered here.

With respect to Figure 8

$$\text{displacement in z-direction} \Big|_{\theta = \bar{\theta}} = w_N \cos \bar{\theta} - u_N \sin \bar{\theta}, \tag{42}$$

and

$$\text{displacement parallel to xy plane} \Big|_{\theta = \bar{\theta}} = w_N \sin \bar{\theta} + u_N \cos \bar{\theta}. \tag{43}$$

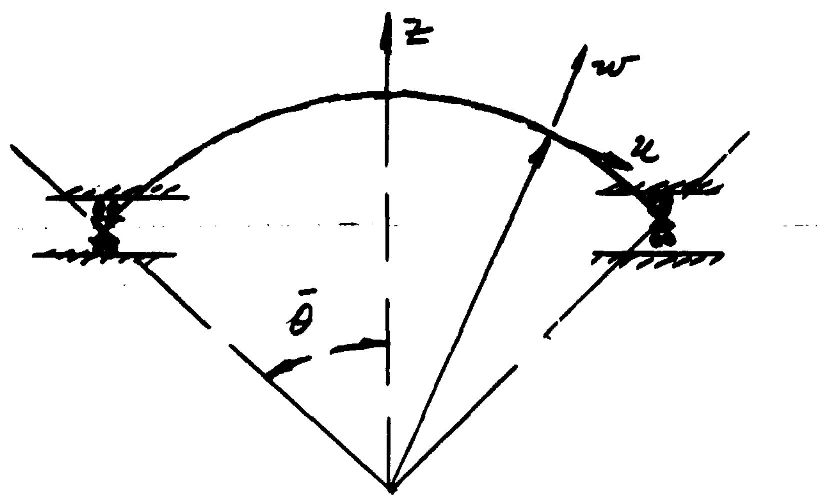

Figure 7

Cross-section view of guided-pinned boundary condition

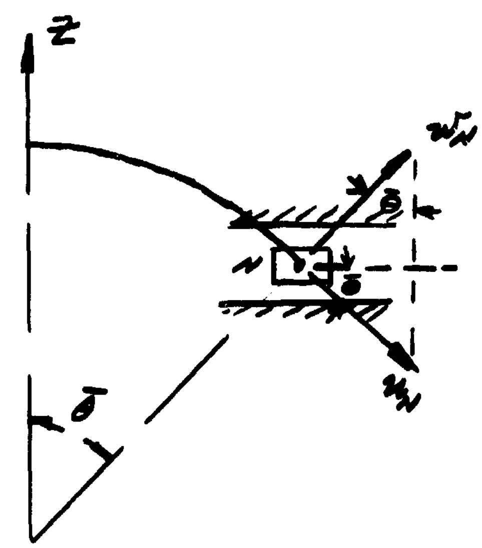

Figure 8

Displacements at guided-pinned edge

Using (41) with (42) results in one mathematical boundary condition,

$$w_N = u_N \tan \bar{\theta} \tag{44}$$

or

$$w_N \cot \bar{\theta} = u_N \quad . \tag{45}$$

Another possible boundary condition is that of the radially guided clamped end. This condition requires that

$$\left.\begin{aligned} &\text{displacement in z-direction} \Big|_{\theta=\bar{\theta}} = 0, \\ &\text{and} \\ &\frac{\partial w}{\partial \theta}\Big|_{\theta=\bar{\theta}} = 0 \quad . \end{aligned}\right\} \tag{46}$$

The first condition again gives rise to either (44) or (45). The second condition by use of (37) gives

$$w_N = \frac{4}{3} w_{N-1} - \frac{1}{3} w_{N-2} \quad . \tag{47}$$

Physically, the boundary condition is somewhere between the clamped and pinned cases of the guided ends; however, how much between is unknown since the exact effects of the epoxy in terms of an equivalent torsional spring effect is unknown.

STRESS RESULTANTS

The stress resultants and hence the stresses in the shell can be determined from the displacements. This is done by the fact that displacements in the shell are known and have been shown to be linear in z. From displacements one can determine strains, and then by use of Hook's law for an isotropic media, one can determine stresses as a function of displacement. The stress resultants are then found by integrating the stresses across the thickness. McDonald [6], who references Vlasov [8], gives the following eqs. for the stress resultants:

$$
\begin{aligned}
M_\theta &= D\left[w_{\theta\theta} + \nu\left(w_{\phi\phi}\csc^2\theta + w_\theta\cot\theta\right)\right],\\
M_\phi &= D\left[\nu w_{\theta\theta} + \left(w_{\phi\phi}\csc^2\theta + w_\theta\cot\theta\right)\right],\\
N_\theta &= K\left[u_\theta + w + \nu\left(u\cot\theta + v_\phi\csc\theta + w\right)\right],\\
N_\phi &= K\left[\nu\left(u_\theta + w\right) + \left(u\cot\theta + v_\phi\cos\theta + w\right)\right],
\end{aligned}
\tag{48}
$$

where M_θ=moment per unit length acting in the plane of a latitude, i.e. in the direction θ-direction,

M_ϕ=moment per unit length acting in ϕ-direction,

N_θ=mid-plane force per unit length acting in θ-direction,

N_ϕ=mid-plane force per unit length acting in ϕ-direction,

$D = \dfrac{Eh^3}{12(1-\nu^2)}$, and $K = \dfrac{Eh}{a(1-\nu^2)}$.

The positive direction of the stress resultants is shown in Fig. 9.

Now substitution of displacements in their assumed series form, eqs. (17) into (48), gives

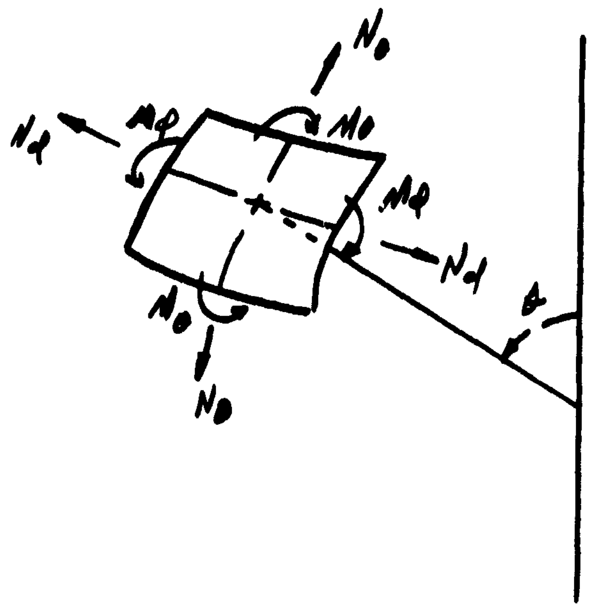

Figure 9

Sign convention of stress resultants

$$M_\theta = D\sum_{m=0}^{\infty}\Big[w_m''(\theta) + \nu\big(-m^2 w_m(\theta)\csc^2\theta + w_m'(\theta)\cot\theta\big)\Big]\cos m_Q ,$$

$$M_Q = D\sum_{m=0}^{\infty}\Big[\nu w_m''(\theta) + \big(-m^2 w_m(\theta)\csc^2\theta + w_m'(\theta)\cot\theta\big)\Big]\cos m_Q ,$$

$$N_\theta = K\sum_{m=0}^{\infty}\Big[u_m'(\theta) + w_m(\theta) + \nu\big(u_m(\theta)\cot\theta + m\, v_m(\theta)\csc(\theta) + w_m(\theta)\big)\Big]\cos m_Q ,$$

$$N_Q = K\sum_{m=0}^{\infty}\Big[\nu\big(u_m'(\theta) + w_m(\theta)\big) + \big(u_m(\theta)\cot\theta + m\, v_m(\theta)\csc\theta + w_m(\theta)\big)\Big]\cos m_Q . \qquad (49)$$

For a spherical cap experiencing symmetry deformation, one considers on the m=0 term so that from (49)

$$M_\theta = D\big[w''(\theta) + \nu w'(\theta)\cot\theta\big],$$

$$M_Q = D\big[\nu w''(\theta) + w'(\theta)\cot\theta\big],$$

$$N_\theta = K\big[u'(\theta) + w(\theta) + \nu\big(u(\theta)\cot\theta + w(\theta)\big)\big],$$

and $$N_Q = K\big[\nu\big(u'(\theta) + w(\theta)\big) + u(\theta)\cot\theta + w(\theta)\big], \qquad (50)$$

where $D = \dfrac{Eh^3}{12(1-\nu^2)}$ and $K = \dfrac{Eh}{a(1-\nu^2)}$.

The stress resultant of the dome point can be written from (48) while noting eqs. [illegible], the boundary conditions at the dome. So that

$$M_\theta\Big|_{\theta=0} = D\, w''(\theta) , \qquad (51)$$

$$M_Q\Big|_{\theta=0} = D\nu\, w''(\theta) ,$$

$$\left.\begin{aligned} N_\theta \Big|_{\theta=0} &= K\left[u'(\theta)+(1+\nu)w(\theta)\right], \\ \text{and} \qquad N_Q \Big|_{\theta=0} &= K\left[\nu\, u'(\theta)+(1+\nu)w(\theta)\right]. \end{aligned}\right\} \quad (52)$$

Using the finite differences approximations and boundary conditions, eqs. (51) and (52) become

$$\left.\begin{aligned} (N_\theta)_{j=0} &= K\left[\frac{\alpha}{2}(u_1-u_0)+(1+\nu)w_0\right] \\ (N_Q)_{j=0} &= K\left[\frac{\nu\alpha}{2}(u_1-u_0)+(1+\nu)w_0\right] \end{aligned}\right\} \quad (53)$$

$$\left.\begin{aligned} (M_\theta)_{j=0} &= \frac{D\alpha^2}{4}(w_0-2w_1+w_2), \\ \text{and} \qquad (M_Q)_{j=0} &= \frac{D\nu\alpha^2}{4}(w_0-2w_1+w_2). \end{aligned}\right\} \quad (54)$$

For an interior point, the stress resultants on the kth segment are given by (48) and to be

$$\begin{aligned} (M_\theta)_{j=k} &= \frac{D}{4}\Big[\alpha^2 w_{k-1} - 2\alpha\left(\alpha+\nu\cot\frac{2k}{\alpha}\right)w_k \\ &\qquad + \alpha\left(\alpha+2\nu\cot\frac{2k}{\alpha}\right)w_{k+1}\Big], \\ (M_Q)_{j=k} &= \frac{D}{4}\Big[\nu\alpha^2 w_{k-1} - 2\alpha\left(\nu\alpha+\cot\frac{2k}{\alpha}\right)w_k \\ &\qquad + \alpha\left(\nu\alpha+2\cot\frac{2k}{\alpha}\right)w_{k+1}\Big], \end{aligned} \quad (55)$$

$$(N_\theta)_{j=k} = \frac{K}{2}\left[\left(2\nu\cot\frac{2k}{\alpha} - \alpha\right)u_k + \alpha u_{k+1} + (1+\nu)w_k\right], \tag{56}$$

and

$$(N_\phi)_{j=k} = \frac{K}{2}\left[\left(2\cot\frac{2k}{\alpha} - \nu\alpha\right)u_k + \nu\alpha u_{k+1} + (1+\nu)w_k\right].$$

Now for the lower edge boundary condition the following finite difference approximations are utilized,

$$q'_N = \frac{\alpha}{2}\left(q_N - q_{N-1}\right),$$

and

$$q''_N = \frac{\alpha^2}{4}\left(q_{N+1} - 2q_N + q_{N-1}\right). \tag{57}$$

This procedure yields finally the following set of equations:

$$\left.\begin{aligned}(M_\theta)_{j=N} &= \frac{D}{4}\Big[\alpha(\alpha - 2\nu\cot\bar{\theta})w_{N-1} - 2\alpha(\alpha - \nu\cot\bar{\theta})w_N + \alpha^2 w_{N+1}\Big],\\ (M_\phi)_{j=N} &= \frac{D}{4}\Big[\alpha(\nu\alpha - 2\cot\bar{\theta})w_{N-1} - 2\alpha(\nu\alpha - \cot\bar{\theta})w_N + \nu\alpha^2 w_{N+1}\Big]\end{aligned}\right\} \tag{58}$$

$$\left.\begin{aligned}(N_\theta)_{j=N} &= \frac{K}{2}\Big[-\alpha u_{N-1} + (\alpha + 2\nu\cot\bar{\theta})u_N + \alpha(1+\nu)w_N\Big],\\ \text{and}\quad (N_\phi)_{j=N} &= \frac{K}{2}\Big[-\nu\alpha u_{N-1} + (\nu\alpha + 2\cot\bar{\theta})u_N + \alpha(1+\nu)w_N\Big].\end{aligned}\right\} \tag{59}$$

DEVELOPMENT OF THE EQUATIONS OF MOTION

Energy Expressions for a General Interior Point

Lagrange's equation may be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \quad , \tag{60}$$

where L = T - V is the Lagrangian equal to the difference between the kinite and potential energy and q_j denotes one of the generalized coordinates $q_1, q_2, \ldots, q_N$. Hence for a stationary system, T=0 and V=V $(q_1, \ldots, q_N, x, y, z)$ so that (( ) reduces to

$$\frac{\partial V}{\partial q_j} = 0 \quad , \tag{61}$$

where q_j denotes $q_0, q_1, \ldots, q_N$ or (N+1) quantities and q is the general displacement denoting u, v and w or 3(N+1) iisplacements in general. The potential energy denoted in (61) is the total potential energy, and thus is the sum of the potential energy of each segment of the dome. Thus,

$$V = \sum_{j=0}^{N} V_j \quad . \tag{62}$$

Noting the types of segments formed in Figure 3 one can write (5) as

$$V = \sum_{j=0}^{N} \left[(v_1)_j + (v_2)_j + \Omega_j \right] . \tag{63}$$

The potential energy in each segment is formed by integrating eqs. (20), (21), and (23) for the interval of each segment and using the appropriate difference approximations for the derivatives as outlined.

Since we are interested in rotational symmetry only, then one can simplify the above equations. Hence considering only m=0 and V=0, then

$$V_1 = \frac{h\mu\pi}{1-\nu}\int_0^{\bar\theta}\Big[(u'+w)^2 + \csc^2\theta(u\cos\theta + w\sin\theta)^2 + 2\nu\csc\theta(u'+w)(u\cos\theta+w\sin\theta)\Big]\sin\theta\,d\theta, \tag{64}$$

$$V_2 = \frac{\mu\pi h^3}{12(1-\nu)a^2}\int_0^{\bar\theta}\Big[(w'')^2 + \csc^4\theta(w'\sin\theta\cos\theta)^2 + 2\nu^2\csc^2\theta(w'')(w'\sin\theta\cos\theta)\Big]\sin\theta\,d\theta,$$

and

$$\_ = -\pi a^2\int_0^{\bar\theta}\Big[\beta_c(X_u + Zw)\Big]\sin\theta\,d\theta, \tag{65}$$

where n is assumed to be constant.

Starting with the dome point and working toward the second boundary, the forward and central difference approximations of O(h) will be utilized, i.e.

$$q_j' = \frac{\alpha}{2}(q_{j+1} - q_j),$$

and

$$q_j'' = \frac{\alpha^2}{4}(q_{j+1} - 2q_j + q_{j-1}) \tag{66}$$

Thus using ... in (6.) one can express the potential energies in each segment. The membrane energy for the jth segment is

$$(V_1)_j = \frac{h\mu\pi}{1-\nu}\int_{\theta_j-1/2}^{\theta_j+1/2}\Big[\Big\{\frac{\alpha}{2}(u_{j+1}-u_j)+w_j\Big\}^2 + \csc^2\theta\Big\{u_j\cos\theta + w_j\sin\theta\Big\}^2 + 2\nu\csc\theta\Big\{\frac{\alpha}{2}(u_{j+1} - u_j) + w_j\Big\}\Big\{u_j\cos\theta + w_j\sin\theta\Big\}\Big]\sin\theta\,d\theta. \tag{67}$$

Collecting terms in ... yields,

$$(V_1)_j = \frac{h\mu\pi}{1-\nu}\int_{\theta_j-1/2}^{\theta_j+1/2}\Big\{\Big[\frac{\alpha^2}{4}(u_{j+1}^2 - 2u_{j+1}u_j + (1+\nu)\alpha u_{j+1}w_j - \alpha u_j w_j + 3w_j^2)\Big]\sin\theta + \Big[2(1+\nu)u_j w_j + \nu\alpha u_{j+1}u_j - \nu\alpha u_j^2\Big]\cos\theta + u_j^2\frac{\cos^2\theta}{\sin\theta}\Big\}d\theta\ .$$

In (68) there are three types of integrals. They are defined as the following:

$$\left.\begin{aligned} I_1 &= \int_{\theta_j-1/\alpha}^{\theta_j+1/\alpha} \sin\theta\, d\theta\,, \\ I_2 &= \int_{\theta_j-1/\alpha}^{\theta_j+1/\alpha} \cos\theta\, d\theta\,, \\ \text{and}\qquad I_3 &= \int_{\theta_j-1/\alpha}^{\theta_j+1/\alpha} \frac{\cos^2\theta\, d\theta}{\sin\theta}\,. \end{aligned}\right\} \tag{69}$$

Evaluating the integrals of (69) yields

$$\left.\begin{aligned} I_1 &= 2\sin\theta_j \sin 1/\alpha\,, \\ I_2 &= 2\cos\theta_j \sin 1/\alpha\,, \\ \text{and}\qquad I_3 &= -2\sin\theta_j \sin 1/\alpha + \ln\tan\frac{\theta_j+1/\alpha}{2} \\ &\quad - \ln\tan\frac{\theta_j-1/\alpha}{2} \end{aligned}\right\} \tag{70}$$

Using (70) in connection with (68), one can thus obtain the membrane energy in the jth segment to be

$$\begin{aligned} (V_1)_j = \frac{h\mu\pi}{1-\nu}\Big\{ & 2\Big[\frac{\alpha^2}{4}(u_{j+1}^2 - 2u_{j+1}u_j + u_j^2) + (1+\nu)(\alpha u_{j+1} w_j \\ & - \alpha u_j w_j + 2w_j^2)\Big]\sin\theta_j \sin 1/\alpha + 2\Big[2(1+\nu)u_j w_j \\ & + \nu\alpha u_{j+1}u_j - \nu\alpha u_j^2\Big]\cos\theta_j \sin 1/\alpha + u_j\Big[-2\sin\theta_j \sin 1/\alpha \\ & + \ln\tan\frac{\theta_j+1/\alpha}{2} - \ln\tan\frac{\theta_j-1/\alpha}{2}\Big]\Big\}\,, \end{aligned} \tag{71}$$

In a similar manner the bending energy for the jth segment was determined and is as follows:

$$(v_2)_j = \frac{\mu \pi h^3}{12(1-\nu)a^2}\Big\{ \frac{\alpha^4}{8}\Big[w_{j+1}^2 - 4w_{j+1}w_j + 2w_{j+1}w_{j-1} + 4w_j^2 - 4w_j w_{j-1} + w_{j-1}^2\Big]\sin\theta_j \sin 1/\alpha + \nu^2 \frac{\alpha^3}{2}\Big[w_{j+1}^2 - 3w_{j+1}w_j + w_{j+1}w_{j-1} + 2w_j^2 - w_j w_{j-1}\Big]\cos\theta_j \sin 1/\alpha + \frac{\alpha^2}{4}\Big[w_{j+1}^2 - 2w_{j+1}w_j + w_j^2\Big]\Big(2\sin\theta_j \sin 1/\alpha + \ln\tan\frac{\theta_j+1/\alpha}{2} - \ln\tan\frac{\theta_j-1/\alpha}{2}\Big) . \tag{72}$$

Observe that the membrane and bending energy expressions give the stiffness matrix coefficients while the external load energy gives the mass matrix by use of D'Alembert's principle. Hence one can now obtain the stiffness matrix coefficients of all points, except the $\bar{\theta}$ edge points and those next to them, by use of eqs. (71), and (72).

Now apply D'Alembert's principle to obtain the equations of motion. Since by D'Alembert's principle

$$\bar{X} = -\rho h \ddot{u} , \quad \bar{Y} = -\rho h \ddot{v} , \quad \bar{Z} = -\rho h \ddot{w} , \tag{73}$$

and assuming steady state conditions,

gives

$$\bar{X} = \rho h \omega^2 u e^{j\omega t} , \quad \bar{Y} = \rho h \omega^2 v e^{j\omega t} , \tag{74}$$

and

$$\bar{Z} = \rho h \omega^2 w e^{j\omega t} .$$

Substitution of (74) into (15) gives for rotationally symmetrical vibrations,

$$\Lambda = -\frac{\pi \rho h a^2 \omega^2}{2}\Big[\int_0^{\bar{\theta}} (u^2 + w^2)\sin\theta \, d\theta\Big] e^{j\omega t} . \tag{75}$$

Now substitution of $u=ue^{jwt}$, etc. into the membrane and bending energies drops the e^{jwt} from all expressions. Hence (75) can be written now for the jth segment as

$$(\Omega)_j = -\pi \rho h a^2 \omega^2 \left[u_j^2 + w_j^2 \right] \sin\theta_j \sin 1/\alpha . \quad (76)$$

Energy Expressions for the Dome Point

One can observe that the two equations of motion for the dome point can be found by evaluating

$$\frac{\partial V}{\partial u_0} = 0 \tag{77}$$

and

$$\frac{\partial V}{\partial w_0} = 0 \quad . \tag{78}$$

First, one considers eq. (78) where the only energies containing u_0 are $(V_1)_{j=0}$ + $(\Omega)_{j=0}$ so that

$$V(u_0) = (V_1)_{j=0} + (\Omega)_{j=0} \ , \tag{79}$$

$$\Rightarrow \frac{\partial V}{\partial u_0} = \frac{\partial (V_1)_{j=0}}{\partial u_0} + \frac{\partial (\Omega)_{j=0}}{\partial u_0} = 0 \ . \tag{80}$$

In equation (78) observe that

$$V(w_0) = (V_1)_{j=0} + (V_2)_{j=0} + (\Omega)_{j=0} + (U_2)_{j=1} \ , \tag{81}$$

Therefore substitution of (81) into (78) yields the remaining eqs. of motion for the dome point given by

$$\frac{\partial V}{\partial w_0} = \frac{\partial (V_1)_{j=0}}{\partial w_0} + \frac{\partial (V_2)_{j=0}}{\partial w_0} + \frac{\partial (\Omega)_{j=0}}{\partial w_0} + \frac{\partial (V_2)_{j=1}}{\partial w_0} = 0 \ . \tag{82}$$

The boundary conditions to be utilized at the dome point are as follows:

$$\left. \begin{aligned} & u(0) = v(0) = 0 \\ \text{and} \quad & \left.\frac{\partial w}{\partial \theta}\right|_{\theta=0} = w'(0) = 0 \ . \end{aligned} \right\} \tag{83}$$

McDonald [6] adds to this set the following boundary condition,

$$\left.\frac{\partial^3 w}{\partial \theta^3}\right|_{\theta=0} = w'''(0) = 0 \tag{84}$$

This is adopted in order to obtain a finite solution at the pole.

Now rewrite the membrane energy of the dome point segment for the interval $0<\theta<1/\alpha$ so that

$$(V_1)_{j=0}=\frac{h\mu\pi}{1-\nu}\int_0^{1/\alpha}\left\{\left[\frac{\alpha}{2}(u_1-u_0)+w_0\right]^2+\cos^2\theta\left[u_0\cos\theta\right.\right.$$
$$\left.+w_0\sin\theta\right]^2+2\nu\cos\theta\left[\frac{\alpha}{2}(u_1-u_0)+w_0\right]\left[u_0\cos\theta\right.$$
$$\left.\left.+w_0\sin\theta\right]\right\}\sin\theta\,d\theta. \qquad (85)$$

Using the boundary condition for rotationally symmetric vibrations, i.e. $u_0=0$, one obtains the membrane energy for the dome segment in this problem to be

$$(V_1)_{j=0}=\frac{h\mu\pi}{1-\nu}\left\{\frac{\alpha^2}{4}u_1^2+\alpha(1+\nu)w_0^2\right.$$
$$\left.+\alpha(1+\nu)w_0u_1\right)\left\}\left\{1-\cos 1/\alpha\right\}. \qquad (86)$$

Likewise for the bending energy,

$$(V_2)_{j=0}=\frac{\mu\pi h^3}{12(1-\nu)a^2}\left[\frac{\alpha^4}{16}\left(w_0^2-4w_0w_1+2w_0w_2+4w_1^2\right.\right.$$
$$\left.\left.-4w_1w_2+w_2^2\right)\right]\left[1-\cos 1/\alpha\right]. \qquad (87)$$

In a similar manner the inertial energy for the dome point results to

$$(\Omega)_{j=0}=-\frac{\pi\rho h a^2}{2}\omega^2\left[w_0^2\right]\left[1-\cos 1/\alpha\right]. \qquad (88)$$

<u>Equation of Motion for the Dome Point</u>

To find the eqs. of motion of the dome point w.r.t. w_0, one uses (82). To do this the expression for $(U_2)_{j=1}$ is needed. To obtain $(U_2)_{j=1}$, it is now possible to make use of the general energy expressions for a jth segment and specialize it for the j=1 point. Before doing this, it seems best to note <u>now</u> that a meridian line is one starting at the dome point and with increasing θ arrives at $\bar{\theta}$, while a longitude is for constant θ and varying ϕ. In Fig. (10) one notes that

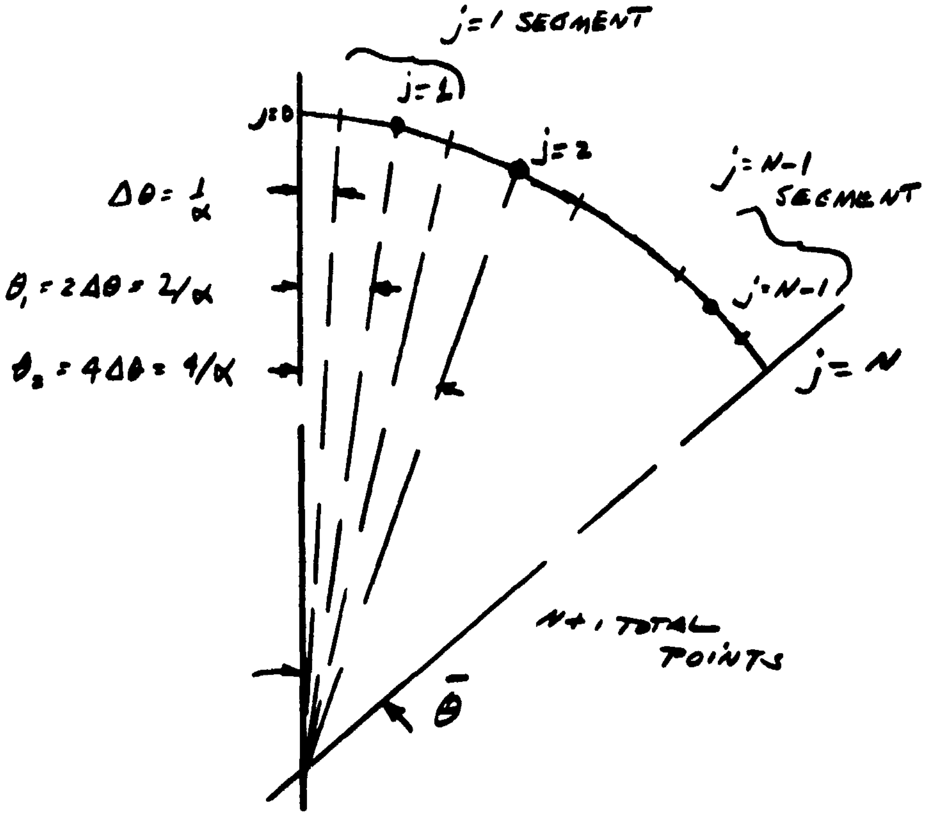

Figure 10

Cross-section view showing segment divisions and their nodes

$$\theta_j = 2j/\alpha \quad . \tag{89}$$

Thus using $\theta_1 = 2/\alpha$, one obtains

$$(V_2)_{j=1} = \frac{\mu \pi h^3}{12(1-\nu)a^2} \Big\{ \frac{\alpha^4}{8} \Big[w_2^2 - 4 w_2 w_1 + 2 w_2 w_0 + 4 w_1^2 - 4 w_1 w_0 + w_0^2 \Big] \sin 2/\alpha \sin 1/\alpha + \frac{\nu^2 \alpha^3}{2} \Big[w_2^2 - 3 w_2 w_1 + w_2 w_0 + 2 w_1^2 - w_1 w_0 \Big] \cos 2/\alpha \sin 1/\alpha + \frac{\alpha^2}{4} \Big[w_2^2 - 2 w_2 w_1 + w_1^2 \Big] (-2 \sin 2/\alpha \sin 1/\alpha + \ln \tan 3/2\alpha - \ln \tan 1/2\alpha) \Big\} \quad . \tag{90}$$

Hence w_0 the eq. of motion of the dome point is as follows:

$$\Big\{ \frac{h \mu \pi}{1-\nu} \big[\alpha (1+\nu) \big] \big[1 - \cos 1/\alpha \big] \Big\} u_1 + \Big\{ 4 \Big(\frac{h \mu \pi}{1-\nu} \Big) (1+\nu) (1 - \cos 1/\alpha) + \frac{\mu \pi h^3}{12(1-\nu) a^2} \Big[\frac{\alpha^4}{16} 2 (1 - \cos 1/\alpha) + \frac{\alpha^4}{8} 2 \sin 2/\alpha \sin 1/\alpha \Big] \Big\} w_0$$
$$+ \Big\{ \frac{\mu \pi h^3}{12(1-\nu) a^2} \Big[\frac{\alpha^4}{16} (-4)(1 - \cos 1/\alpha) + \frac{\alpha^4}{8} (-4) \sin 2/\alpha \sin 1/\alpha - \frac{\nu^2 \alpha^3}{2} \cos 2/\alpha \sin 1/\alpha \Big] \Big\} w_1$$
$$+ \Big\{ \frac{\mu \pi h^3}{12(1-\nu) a^2} \Big[\frac{\alpha^4}{16} (2)(1 - \cos 1/\alpha) + \frac{\alpha^4}{8} (2) \sin 2/\alpha \sin 1/\alpha + \frac{\nu^2 \alpha^3}{2} \cos 2/\alpha \sin 1/\alpha \Big] \Big\} w_2$$
$$- \Big\{ \pi \rho h a^2 \omega^2 (1 - \cos 1/\alpha) \Big\} w_0 = 0 . \tag{91}$$

The final stiffness coefficients are defined similar to McDonald's [9] i.e., S_{ij}^{kl} where $\frac{l}{j}$ refer to the displacement component by which the coefficient is multiplied and $\frac{k}{i}$ designates the particular eqs. from which the coefficient comes. Hence (91) can be written as

$$(S^{w\,u}_{0\,1})u_1+(S^{w\,w}_{0\,0})w_0+(S^{w\,w}_{0\,1})\,w_1$$
$$+(S^{w\,w}_{0\,2})w_2-(m^{w\,w}_{0\,0})w_0=0\;,$$

where
$$S^{w\,u}_{0\,1}=\frac{h\mu\pi}{1-\nu}\left[\alpha(1+\nu)\right]\left[1-\cos 1/\alpha\right],$$
$$S^{w\,u}_{0\,0}=4\left(\frac{h\mu\pi}{1-\nu}\right)(1+\nu)(1-\cos 1/\alpha)+\frac{\mu\pi h^3}{12(1-\nu)a^2}\left[\frac{\alpha^4}{8}(1-\cos 1/\alpha)\right.$$
$$\left.+\frac{\alpha^4}{4}\sin 3/\alpha\,\sin 1/\alpha\right], \tag{92}$$

$$S^{w\,w}_{0\,1}=-\frac{\mu\pi h^2}{12(1-\nu)a^2}\left[\frac{\alpha^4}{4}(1-\cos 1/\alpha)+\frac{\alpha^4}{2}\sin 2/\alpha\,\sin 1/\alpha\right.$$
$$\left.+\frac{\nu^2\alpha^3}{2}\cos 2/\alpha\,\cos 1/\alpha\right],$$

$$S^{w\,w}_{0\,2}=\frac{\mu\pi h^3}{12(1-\nu)a^2}\left[\frac{\alpha^4}{8}(1-\cos 1/\alpha)+\frac{\alpha^4}{4}\sin 2/\alpha\,\sin 1/\alpha\right.$$
$$\left.+\frac{\nu^2\alpha^3}{2}\cos 2/\alpha\,\sin 1/\alpha\right],$$

and
$$m^{w\,w}_{0\,0}=\pi\rho h a^2\omega^2(1-\cos 1/\alpha).$$

Equation of Motion for j=1 Point

The eqs. of motion for the point next to the dome point will now be obtained since the eqs. w.r.t. the u_1 and w_1 displacements cannot be obtained from a general expressed. This is due to the fact that the energy expressions for j=0 contain u_1 and w_1.

The eqs. w.r.t. u_1 will be considered first. The energy which is a function of u_1 is

$$V(u_1)=(V_1)_{j=0}+(V_1)_{j=1}+(\Omega)_{j=1}\;, \tag{93}$$

so that the eqs. of motion is given by

$$\frac{\partial V}{\partial u_1} = \frac{\partial (U_1)_{j=0}}{\partial u_1} + \frac{\partial (U_1)_{j=1}}{\partial u_1} + \frac{\partial (\Omega)_{j=1}}{\partial u_1} = 0 \; . \qquad (94)$$

Substitution of $U_{j=0,1}$ and $\Omega_{j=1}$ into equation (94) and evaluating the partial derivatives yields,

$$(S_{1\,1}^{u\,u})\, u_1 + (S_{1\,2}^{u\,u})\, u_2 + (S_{1\,0}^{u\,w})\, w_0 + (S_{1\,1}^{u\,w})\, w_1 - (m_{1\,1}^{u\,u})\, u_1 = 0 \; ,$$

where

$$S_{1\,1}^{u\,u} = \frac{h\mu\pi}{1-\nu}\Big[\frac{\alpha^2}{2}(1-\cos 1/\alpha) + \alpha^2 \sin 2/\alpha \sin 1/\alpha - 4\nu\alpha \cos 2/\alpha \sin 1/\alpha - 4 \sin 2/\alpha \sin 1/\alpha + 2 \ln \tan 3/2\,\alpha - 2 \ln \tan 1/2\alpha\Big],$$

$$S_{1\,2}^{u\,u} = \frac{h\mu\pi}{1-\nu}\Big[-\alpha^2 \sin 3/\alpha \sin 1/\alpha + 2\nu\alpha \cos 3/\alpha \sin 1/\alpha\Big],$$

$$S_{1\,0}^{u\,u} = \frac{h\mu\pi}{1-\nu}\Big[\alpha(1+\nu)(1-\cos 1/\alpha)\Big] \; , \qquad (95)$$

$$S_{1\,1}^{u\,w} = \frac{h\mu\pi}{1-\nu}\Big[-2\alpha(1+\nu)\sin 2/\alpha \sin 1/\alpha + 4(1+\nu)\cos 2/\alpha \sin 1/\alpha\Big] \; ,$$

and

$$m_{1\,1}^{u\,u} = 2\pi \rho h a^2 \omega^2 \sin 2/\alpha \sin 1/\alpha \; .$$

Equation ( ) is the u_1 eq . of motion of the point $j=1$.

To obtain the 2nd eq. of motion for the point $j=1$, one obtains the energy expressions which are functions of w_1. Hence

$$V(w_1) = (U_1)_{j=1} + (U_2)_{j=0} + (U_2)_{j=1} + (U_2)_{j=2} + (\Omega)_{j=1} \qquad (96)$$

so that

$$\frac{\partial V}{\partial w_1} = \frac{\partial (V_1)_{j=1}}{\partial w_1} + \frac{\partial (U_2)_{j=0}}{\partial w_1} + \frac{\partial (V_2)_{j=1}}{\partial w_1} + \frac{\partial (U_2)_{j=2}}{\partial w_1} + \frac{\partial (\Omega)_{j=1}}{\partial w_1} = 0 \; . \tag{97}$$

In a similar manner the equation of motion of the j=1 point with respect to w_1 was determined to be,

$$\left(S_{1\;1}^{w\,u}\right)u_1 + \left(S_{1\;2}^{w\,u}\right)u_2 + \left(S_{1\;0}^{w\,w}\right)w_0 + \left(S_{1\;1}^{w\,w}\right)w_1 + \left(S_{1\;2}^{w\,w}\right)w_2 + \left(S_{1\;3}^{w\,w}\right)w_3 - \left(m_{1\;1}^{w\,w}\right)w_1 = 0,$$

where

$$S_{1\;1}^{w\,u} = \frac{h\mu\pi}{1-\nu}\left[-2\alpha(1+\nu)\sin 3/\alpha \sin 1/\alpha + 4(1+\nu)\cos 2/\alpha \sin 1/\alpha\right]$$

$$S_{1\;2}^{w\,u} = \frac{h\mu\pi}{1-\nu}\left[2\alpha(1+\nu)\sin 2/\alpha \sin 1/\alpha\right],$$

$$S_{1\;0}^{w\,w} = -\frac{\mu\pi h^3}{12(1-\nu)\alpha^2}\left[\frac{\alpha^4}{4}(1-\cos 1/\alpha) + \frac{\alpha^4}{2}\sin 2/\alpha \sin 1/\alpha + \frac{\nu^2\alpha^3}{2}\cos 2/\alpha \sin 1/\alpha\right],$$

$$S_{1\;1}^{w\,w} = \frac{h\mu\pi}{1-\nu}\left[8(1+\nu)\sin 2/\alpha \sin 1/\alpha\right] + \frac{\mu\pi h^3}{12(1-\nu)\alpha^2}\left[\frac{\alpha^4}{2}(1-\cos 1/\alpha) + \alpha^4 \sin 2/\alpha \sin 1/\alpha + \frac{\alpha^4}{4}\sin 4/\alpha \sin 1/\alpha + 2\nu^2\alpha^3\cos 4/\alpha \sin 1/\alpha + \frac{\alpha^2}{2}\left(2\alpha \sin 2/\alpha \sin 1/\alpha + \ln\tan 3/2\alpha - \ln\tan 1/2\alpha\right)\right], \tag{98}$$

$$S_{1\;2}^{w\,w} = -\frac{\mu\pi h^3}{12(1-\nu)\alpha^2}\left[\frac{\alpha^4}{4}(1-\cos 1/\alpha) + \frac{\alpha^4}{2}\sin 2/\alpha \sin 1/\alpha + \frac{\alpha^4}{2}\sin 4/\alpha \sin 1/\alpha + \frac{\nu^2\alpha^3}{2}\cos 2/\alpha \sin 1/\alpha + \frac{\nu^2\alpha^3}{2}\cos 4/\alpha \sin 1/\alpha + \frac{\alpha^2}{2}\left(2\alpha \sin 2/\alpha \sin 1/\alpha + \ln\tan 3/2\alpha - \ln\tan 1/2\alpha\right)\right],$$

$$S_{1\,3}^{w\,w} = \frac{\mu \pi h^3}{12(1-\nu)a^2}\left[\frac{\alpha^4}{4} \sin 4/\alpha \, \sin 1/\alpha + \frac{\nu^2 \alpha^3}{2} \cos 4/\alpha \, \sin 1/\alpha\right],$$

and

$$m_{1\,1}^{w\,w} = 2\pi \rho h a^2 \omega^2 \sin 3/\alpha \, \sin 1/\alpha \, .$$

Equations of Motion for j=2 Point

Now to obtain the w_2 eq. of motion, one forms the energies containing the factor w_2. Therefore,

$$V(w_2) = (U_1)_{j=2} + (U_2)_{j=0} + (U_2)_{j=1} + (U_2)_{j=2} + (U_2)_{j=3} + (\Omega)_{j=2} \quad . \qquad (99)$$

Thus the w_2 eqs. of motion is given by

$$\frac{\partial (U_1)_{j=2}}{\partial w_2} + \frac{\partial (U_2)_{j=0}}{\partial w_2} + \frac{\partial (U_2)_{j=1}}{\partial w_2} + \frac{\partial (U_2)_{j=2}}{\partial w_2} + \frac{\partial (U_2)_{j=3}}{\partial w_2} + \frac{\partial (\Omega)_{j=2}}{\partial w_2} = 0 \qquad (100)$$

Again substituting for the various energy expressions yields the following w_2 eq. of motion for the j=2 point.

$$\begin{aligned}(S^{wu}_{22})u_2 + (S^{wu}_{23})u_3 + (S^{ww}_{20})w_0 + (S^{ww}_{21})w_1 \\ + (S^{ww}_{22})w_2 + (S^{ww}_{23})w_3 + (S^{ww}_{24})w_4 \\ - (m^{ww}_{22})w_2 = 0,\end{aligned}$$

where

$$S^{wu}_{22} = \frac{h\mu\pi}{1-\nu}\left[-2\alpha(1+\nu)\sin 4/\alpha \sin 1/\alpha + 4(1+\nu)\cos 4/\alpha \sin 1/\alpha\right],$$

$$S^{wu}_{23} = \frac{h\mu\pi}{1-\nu}\left[2\alpha(1+\nu)\sin 4/\alpha \sin 1/\alpha\right], \qquad (101)$$

$$S^{ww}_{20} = \frac{\mu\pi h^3}{12(1-\nu)\alpha^2}\left[\frac{\alpha^4}{8}(1-\cos 4/\alpha) + \frac{\alpha^4}{4}\sin 2/\alpha \sin 1/\alpha + \frac{3\nu\alpha^3}{2}\cos 2/\alpha \sin 1/\alpha\right],$$

$$\begin{aligned}S^{ww}_{21} = -\frac{\mu\pi h^3}{12(1-\nu)\alpha^2}\Big[&\frac{\alpha^4}{4}(1-\cos 4/\alpha) + \frac{\alpha^4}{2}\sin 2/\alpha \sin 1/\alpha + \frac{\alpha^4}{2}\sin 4/\alpha \sin 1/\alpha \\ &+ \frac{3\nu\alpha^3}{2}\cos 2/\alpha \sin 1/\alpha + \frac{\nu\alpha^3}{2}\cos 4/\alpha \sin 1/\alpha \\ &+ \frac{\alpha^2}{2}\left(\alpha \sin 2/\alpha \sin 1/\alpha + \ln \tan 3/2\alpha - \ln \tan 1/2\alpha\right)\Big],\end{aligned}$$

$$S_2^{w}{}_2^{w} = \frac{h\mu\pi}{1-\nu}\left[8(1+\nu)\sin 4/\alpha \sin 1/\alpha\right] + \frac{\mu\pi h^3}{12(1-\nu)a^2}\left[\frac{\alpha^4}{8}(1-\cos 1/\alpha)\right.$$

$$+\frac{\alpha^4}{4}\sin 2/\alpha \sin 1/\alpha + \alpha^4 \sin 4/\alpha \sin 1/\alpha + \frac{\alpha^4}{4}\sin 6/\alpha \sin 1/\alpha$$

$$+\nu^2\alpha^3\cos 2/\alpha \sin 1/\alpha + 2\nu^2\alpha^3\cos 4/\alpha \sin 1/\alpha + \frac{\alpha^2}{2}(2\sin 2/\alpha \sin 1/\alpha$$

$$+\ln\tan 3/2\alpha - \ln\tan 1/2\alpha) + \frac{\alpha^2}{2}(2\sin 4/\alpha \sin 1/\alpha + \ln\tan 3/2\alpha$$

$$\left.-\ln\tan 3/2\alpha)\right],$$

$$S_2^{w}{}_3^{w} = \frac{-\mu\pi h^3}{12(1-\nu)a^2}\left[\frac{\alpha^4}{2}\sin 4/\alpha \sin 1/\alpha + \frac{\alpha^4}{2}\sin 6/\alpha \sin 1/\alpha\right.$$

$$+\frac{3\nu^2\alpha^3}{2}\cos 4/\alpha \sin 1/\alpha + \frac{\nu^2\alpha^3}{2}\cos 6/\alpha \sin 1/\alpha + \frac{\alpha^2}{2}(2\sin 4/\alpha \sin 1/\alpha$$

$$\left.+\ln\tan 5/2\alpha - \ln\tan 3/2\alpha\right],$$

$$S_2^{w}{}_4^{w} = \frac{\mu\pi h^3}{12(1-\nu)a^2}\left[\frac{\alpha^4}{4}\sin 6/\alpha \sin 1/\alpha + \frac{\nu^2\alpha^3}{2}\cos 6/\alpha \sin 1/\alpha\right],$$

and
$$m_2^{w}{}_2^{w} = 2\pi\rho h a^2 w^2 \sin 4/\alpha \sin 1/\alpha.$$

It should be noted that the u_2 equation of motion can be obtained from the general interior equations since it is not affected by the boundary point.

Equations of Motion for the General j=k Interior Points

Now the eqs. of motion for a general interior point are needed. These will be obtained by considering the partial derivatives

$$\frac{\partial V}{\partial u_K} = 0 \tag{102}$$

and

$$\frac{\partial V}{\partial w_K} = 0 \tag{103}$$

For (102) the energies containing the factor u_k are the only ones of interest; hence,

$$V(u_K) = (U_1)_{j=K-1} + (U_1)_{j=K} + (\Omega)_{j=K} \quad . \tag{104}$$

Thus, substituting (104) into (102) yields,

$$\frac{\partial(U_1)}{\partial u_K}_{j=K-1} + \frac{\partial(U_1)}{\partial u_K}_{j=K} + \frac{\partial(\Omega)}{\partial u_K}_{j=K} = 0 \qquad (105)$$

The energies containing the factor w_K are as follows:

$$V(w_K) = (U_1)_{j=K} + (U_2)_{j=K-1} + (U_2)_{j=K} + (U_2)_{j=K+1} (\Omega)_{j=K}. \qquad (106)$$

Using (1[illegible]) in ([illegible]) gives

$$\frac{\partial(U_1)}{\partial w_K}_{j=K} + \frac{\partial(U_2)}{\partial w_K}_{j=K-1} + \frac{\partial(U_2)}{\partial w_K}_{j=K} + \frac{\partial(U_2)}{\partial w_K}_{j=K+1} + \frac{\partial(\Omega)}{\partial w_K}_{j=K} = 0. \qquad (107)$$

Now to obtain one of the general equations of motion for the j=k point, the various energies defined in equation (106 are substituted into eqaution (107). This yields the u equation of motion for the j=k point as follows:

$$(S^{uu}_{K\,K-1})u_{K-1} + (S^{uu}_{KK})u_K + (S^{uu}_{K\,K+1})u_{K+1} + (S^{uw}_{K\,K=1})w_{K-1} + (S^{uw}_{KK})w_K - (m^{uu}_{KK})u_K = 0 ,$$

where $S^{uu}_{K\,K-1} = \frac{h\mu\pi}{1-\nu}\left[-\alpha^2 \sin\frac{2(K-1)}{\alpha}\sin 1/\alpha + 2\nu\alpha\cos\frac{2(h-1)}{\alpha}\sin 1/\alpha\right],$

$$\begin{aligned} S^{uu}_{KK} = \frac{h\mu\pi}{1-\nu}\Big[&\alpha^2 \sin\frac{2(K-1)}{\alpha}\sin 1/\alpha + \alpha^2 \sin\frac{2h}{\alpha}\sin 1/\alpha \\ &- 4\nu\alpha\cos\frac{2h}{\alpha}\sin 1/\alpha + 2(2\sin\frac{2h}{\alpha}\sin 1/\alpha \\ &+ \ln\tan\frac{2(K+1)}{2\alpha} - \ln\tan\frac{\alpha K-1}{2\alpha})\Big], \end{aligned} \qquad (108)$$

$$S^{uu}_{K\,K+1} = \frac{h\mu\pi}{1-\nu}\left[-\alpha^2 \sin\frac{2K}{\alpha}\sin 1/\alpha + 2\nu\alpha\cos\frac{2K}{\alpha}\sin 1/\alpha\right],$$

$$S^{uw}_{K\,K-1} = \frac{h\mu\pi}{1-\nu}\left[2\alpha(1+\nu)\sin\frac{2(K-1)}{\alpha}\sin 1/\alpha\right]$$

$$S^{ww}_{kk} = \frac{h\mu\pi}{1-\nu}\left[-2\alpha(1+\nu)\sin\frac{2k}{\alpha}\sin 1/\alpha + 4(1+\nu)\cos\frac{2k}{\alpha}\sin 1/\alpha\right],$$

and $M^{uu}_{KK} = 2\pi\rho h a^2 \omega^2 \sin\frac{2K}{\alpha}\sin 1/\alpha$.

The second equation of motion is found by following the general procedures outlined above. This yields the equation of motion for w at the j=k point, i. e.

$$\begin{aligned}
&(S^{wu}_{KK})u_K + (S^{w\ u}_{K\ K+1})u_{K+1} + (S^{w\ w}_{K\ K-2})w_{K-2} \\
&+(S^{w\ w}_{K\ K-1})w_{K-1} + (S^{w\ w}_{K\ K})w_K + (S^{w\ w}_{K\ K+1})w_{K+1} \\
&+(S^{w\ w}_{K\ K+2})w_{K+2} - (M^{w\ w}_{K\ K})w_K = 0 ,
\end{aligned}$$

where $S^{w\ u}_{k\ k} = \frac{h\mu\pi}{1-\nu}\left[-2\alpha(1+\nu)\sin\frac{2k}{\alpha}\sin 1/\alpha + 4(1+\nu)\cos\frac{2k}{\alpha}\sin 1/\alpha\right],$

$$S^{w\ u}_{k\ k+1} = \frac{h\mu\pi}{1-\nu}\left[2\alpha(1+\nu)\sin\frac{2k}{\alpha}\sin 1/\alpha\right], \tag{109}$$

$$S^{w\ w}_{k\ k-2} = \frac{\mu\pi h^3}{12(1-\nu)\alpha^2}\left[\frac{\alpha^4}{4}\sin\frac{2(k-1)}{\alpha}\sin 1/\alpha + \frac{\nu^2\alpha^3}{2}\cos\frac{2(k-1)}{\alpha}\sin 1/\alpha\right],$$

$$\begin{aligned}
S^{w\ w}_{K\ k-1} = -\frac{\mu\pi h^3}{12(1-\nu)\alpha^2}\Bigg[&\frac{\alpha^4}{2}\sin\frac{2(K-1)}{\alpha}\sin 1/\alpha + \frac{\alpha^4}{2}\sin\frac{2k}{\alpha}\sin 1/\alpha \\
&+ \frac{3\nu^2\alpha^3}{2}\cos\frac{2(K-1)}{\alpha}\sin 1/\alpha + \frac{\nu^2\alpha^3}{2}\cos\frac{2K}{\alpha}\sin 1/\alpha \\
&+ \frac{\alpha^2}{2}\left(2\sin\frac{2(K-1)}{\alpha}\sin 1/\alpha + \ln\tan\frac{2K-1}{2\alpha} - \ln\tan\frac{2K-3}{2\alpha}\right)\Bigg],
\end{aligned}$$

$$\begin{aligned}
S^{w\ w}_{k\ K} = \frac{\mu\pi h}{1-\nu}\left[8(1+\nu)\sin\frac{2k}{\alpha}\sin 1/\alpha\right] + \frac{\mu\pi h^3}{12(1-\nu)\alpha^2}\Bigg[&\frac{\alpha^4}{4}\sin\frac{2(K-1)}{\alpha}\sin 1/\alpha \\
&+ \alpha^4\sin\frac{2K}{\alpha}\sin 1/\alpha + \frac{\alpha^4}{4}\sin\frac{2(K+1)}{2}\sin 1/\alpha \\
&+ \nu^2\alpha^3\cos\frac{2(K-1)}{\alpha}\sin 1/\alpha + 2\nu^2\alpha^3\cos\frac{2K}{\alpha}\sin 1/\alpha
\end{aligned}$$

$$+\frac{\alpha^2}{2}\left(-2\sin\frac{2(k-1)}{\alpha}\sin 1/\alpha + \ln\tan\frac{2k-1}{2\alpha} - \ln\tan\frac{2k-3}{2\alpha}\right)$$

$$+\frac{\alpha^2}{2}\left(-2\sin\frac{2k}{\alpha}\sin 1/\alpha + \ln\tan\frac{2k-3}{2\alpha} + \ln\tan\frac{2k+1}{2\alpha}\right.$$

$$\left.+\frac{\alpha^2}{2}\left(-2\sin\frac{2k}{\alpha}\sin 1/\alpha + \ln\tan\frac{2k+1}{2\alpha} - \ln\tan\frac{2k-1}{2\alpha}\right)\right],$$

$$S^{ww}_{k\,k+2} = \frac{\mu\pi h^3}{12(1-\nu)a^2}\left[\frac{\alpha^4}{4}\sin\frac{2(k+1)}{\alpha}\sin 1/\alpha + \frac{\nu^2\alpha^3}{2}\cos\frac{2(k+1)}{\alpha}\sin 1/\alpha\right],$$

$$m^{ww}_{k\,k} = 2\pi\rho h a^2 w^2 \sin\frac{2k}{\alpha}\sin 1/\alpha$$

$$S^{ww}_{k\,k+1} = \frac{-\mu\pi h^3}{12(1-\nu)a^2}\left[\frac{\alpha^4}{2}\sin\frac{2k}{\alpha}\sin 1/\alpha + \frac{\alpha^4}{2}\sin\frac{2(k+1)}{\alpha}\sin 1/\alpha\right.$$
$$+ \frac{3\nu^2\alpha^3}{2}\cos\frac{2k}{\alpha}\sin 1/\alpha + \frac{\nu^2\alpha^3}{2}\cos\frac{2(k+1)}{\alpha}\sin 1/\alpha$$
$$\left.+\frac{\alpha^2}{2}\left(-2\sin\frac{2k}{\alpha}\sin 1/\alpha + \ln\tan\frac{2k+1}{2\alpha} - \ln\tan\frac{2k-1}{2\alpha}\right)\right]$$

Energy Expression for the Lower Edge Half-Segment

The energy expressions for the j=N half-segment are

$$(V_1)_{j=N} = \frac{h\mu\pi}{1-\nu}\int_{\bar{\theta}-1/\alpha}^{\bar{\theta}}\Big[(u_N' + w_N)^2 + \csc^2\theta(u_N\cos\theta \tag{110}$$

$$+w_{\text{N}}\sin\theta)^2 + 2\nu\csc\theta(u_N' + w_N)(u_N\cos\theta + w_N\sin\theta)\Big]\sin\theta\, d\theta,$$

$$(V_2)_{j=N} = \frac{\mu\pi h^3}{12(1-\nu)a^2}\int_{\bar{\theta}-1/\alpha}^{\bar{\theta}}\Big[(w_N'')^2 + \csc^4\theta(w_N'\sin\theta\cos\theta)^2$$

$$+2\nu^2\csc^2\theta(w_N'')(w'\sin\theta\cos\theta)\Big]\sin\theta\, d\theta,$$

and

$$(\Omega)_{j=N} = -2\pi\rho h a^2\omega^2\int_{\bar{\theta}-1/\alpha}^{\bar{\theta}}\left[u_N^2 + w_N^2\right]\sin\theta\, d\theta. \tag{111}$$

For the lower edge, the finite difference approximations to be used are

$$g_N' = \frac{\alpha}{2}(g_N - g_{N-1})$$

and

$$g_N'' = \frac{\alpha^2}{4}(g_{N+1} - 2g_N + g_{N-1}). \tag{112}$$

Substituting the finite difference approximations and evaluating yields,

$$(V_1)_{j=N} = \frac{h\mu\pi}{1-\nu}\int_{\bar{\theta}-1/\alpha}^{\bar{\theta}}\Big\{\Big[\frac{\alpha^2}{4}(u_N^2 - 2u_N u_{N-1} + u_{N-1}^2)$$

$$+\alpha(1+\nu)(u_N w_N - u_{N-1}w_N) + 2(1+\nu)w_N^2\Big]\sin\theta \tag{113}$$

$$+\left[2(1+\nu)u_N w_N + \nu\alpha(u_N^2 - u_N u_{N-1})\right]\cos\theta$$

$$+u_N^2\frac{\cos^2\theta}{\sin\theta}\Big\}\, d\theta \quad ,$$

$$(U_2)_{j=N} = \frac{\mu\pi h^3}{12(1-\nu)\alpha^2}\int_{\bar\theta-1/\alpha}^{\bar\theta}\Big\{\Big[\frac{\alpha^4}{16}(w_{N+1}^2 - 4\,w_{N+1}\,w_N + 2w_{N+1}w_{N-1}$$

$$+4\,w_N^2 - 4\,w_N\,w_{N-1} + w_{N-1}^2)\Big]\sin\theta + \Big[\frac{\nu^2\alpha^3}{4}(w_{N+1}w_N$$

$$-w_{N+1}w_{N-1} - 2w_N^2 + 3\,w_N w_{N-1} - w_{N-1}^2\Big]\cos\theta + \Big[\frac{\alpha^2}{4}(w_N^2 \qquad (114)$$

$$-2w_N w_{N-1} + w_{N-1}^2)\Big]\frac{\cos^2\theta}{\sin\theta} \quad d\theta \quad .$$

Eqs. (113) and (114) are the energies of the lower edge half-segment before the boundary conditions are applied. The integrals in (113) and (114) can be evaluated as was done previously. Evaluating the integrals yields the following energy expressions:

$$(U_1)_{j=N} = \frac{h\mu\pi}{1-\nu}\Big\{\Big[\frac{\alpha^2}{4}(u_N^2 - 2u_N\,u_{N-1} + u_{N-1}^2) + \alpha(1+\nu)u_N w_N$$

$$-u_{N-1}w_N) + 2(1+\nu)w_N^2\Big]\Big[-(1-\cos 1/\alpha)\cos\bar\theta + \sin\bar\theta\sin 1/\alpha\Big]$$

$$+\Big[2(1+\nu)\,u_N w_N + \nu\alpha(u_N^2 - u_N u_{N-1})\Big]\Big[(1-\cos 1/\alpha)\sin\bar\theta + \cos\bar\theta\sin 1/\alpha\Big]$$

$$+u_N^2\Big[(1-\cos 1/\alpha)\cos\bar\theta - \sin\bar\theta\sin 1/\alpha + \ln\tan\frac{\bar\theta}{2} - \ln\tan\frac{2N-1}{2\alpha}\Big], \qquad (115)$$

$$(U_2)_{j=N} = \frac{\mu\pi h^3}{12(1-\nu)\alpha^2}\Big\{\Big[\frac{\alpha^4}{16}(w_{N+1}^2 - 4\,w_{N+1}w_N + 2w_{N+1}w_{N-1} + 4w_N^2$$

$$-4w_N w_{N-1} + w_{N-1}^2)\Big]\Big[-(1-\cos 1/\alpha)\cos\bar\theta + \sin\bar\theta\sin 1/\alpha\Big] \qquad (116)$$

$$+\left[\frac{\nu^2-3}{4}\left(w_{N+1}w_N - w_{N+1}w_{N-1} - 2w_N^2 + 3w_N w_{N-1} - w_{N-1}^2\right)\right]\left[(1 - \cos 1/\alpha)\sin\bar{\theta} + \cos\bar{\theta}\sin 1/\alpha\right] + \left[\frac{\alpha^2}{4}\left(w_N^2 - 2w_N w_{N-1} + w_{N-1}^2\right)\right]\left[(1-\cos 1/\alpha)\cos\bar{\theta} - \sin\bar{\theta}\sin 1/\alpha + \ln\tan\bar{\theta}/2 - \ln\tan\frac{2N-1}{2\alpha}\right]\Bigg\} \quad .$$

Up to this point, all work on the (j=N) lower edge half-segment has been general and can be used for any edge condition. In order to obtain the eqs. of motion for a particular b. c., one must, now specialize (115) and (116)by use of the appropriate boundary condition.

The inertial energy term for the j=N point is now

$$(\Omega)_{j=N} = -\pi\rho h a^2 \omega^2 \left[u_N^2 + w_N^2\right]\left[-(1-\cos 1/\alpha)\cos\bar{\theta} + \sin\bar{\theta}\sin 1/\alpha\right] \tag{117}$$

Equation of Motion for the j=N Point

The b. c. for the guided-pinned case are given by (40) and (118) will be used in eqs. (115)and (116)while (40) will be saved. Eqs. (118) is

$$w_{N+1} = \frac{2}{\alpha}(\alpha - \nu\cot\bar{\theta})w_N - 1/\alpha(\alpha - 2\nu\cot\bar{\theta})w_{N-1} \quad . \tag{118}$$

Therefore, the bending energy for the pinned boundary edge half-segment is given by equation (116) i.e.

$$(U_2)_{j=N} = \frac{\mu\pi h^3}{12(1-\nu)a^2}\Bigg\{\left[\frac{\alpha^2}{4}(\alpha - \nu\cot\bar{\theta})^2 u_N^2 \cot^2\bar{\theta} - \frac{\alpha^2}{4}(\alpha - \nu\cot\bar{\theta})(\alpha - 2\nu\cot\bar{\theta})w_{N-1}u_N\cot\bar{\theta} + \frac{\alpha^2}{16}(\alpha - 2\nu\cot\bar{\theta})^2 w_{N-1}^2\right.$$

$$-\frac{\alpha^3}{2}(\alpha-\nu\cot\bar\theta)u_N^2\cot^2\bar\theta+\frac{\alpha^3}{4}(\alpha-2\nu\cot\bar\theta)w_{N-1}u_N\cot\bar\theta$$

$$+\frac{\alpha^3}{4}(\alpha-\nu\cot\bar\theta)w_{N-1}u_N\cot\bar\theta-\frac{\alpha^3}{8}(\alpha-2\nu\cot\bar\theta)w_{N-1}^2$$

$$+\frac{\alpha^4}{16}(4u_N^2\cot^2\bar\theta-4w_{N-1}u_N\cot\bar\theta+w_{N-1}^2)\Big]\Big[\sin\bar\theta\sin 1/\alpha$$

$$-(1-\cos 1/\alpha)\cos\bar\theta\Big]+\Big[\frac{\nu^2\alpha^2}{2}(\alpha-\nu\cot\bar\theta)u_N^2\cot^2\bar\theta$$

$$-\frac{\nu^2\alpha^2}{4}(\alpha-2\nu\cot\bar\theta)w_{N-1}u_N\cot\bar\theta-\frac{\nu^2\alpha^2}{2}(\alpha-\nu\cot\bar\theta)w_{N-1}u_N\cot\bar\theta$$

$$+\frac{\nu^2\alpha^2}{4}(\alpha-2\nu\cot\bar\theta)w_{N-1}^2+\frac{\nu^2\alpha^3}{4}(-2u_N^2\cot^2\bar\theta+3w_{N-1}u_N\cot\bar\theta$$

$$-w_{N-1}^2)\Big]\Big[\cos\bar\theta\sin 1/\alpha+(1-\cos 1/\alpha)\sin\bar\theta\Big]+\Big[\frac{\alpha^2}{4}(u_N^2\cot^2\bar\theta$$

$$-2w_{N-1}u_N\cot\bar\theta+w_{N-1}^2)\Big]\Big[(1-\cos 1/\alpha)\cos\theta-\sin\bar\theta\sin 1/\alpha$$

$$+\ln\tan\frac{\bar\theta}{2}-\ln\tan\frac{2N-1}{2\alpha}\Big] \quad . \qquad (119)$$

Now noting eq. 40 to be

$$w_N=u_N\tan\bar\theta, \qquad (120)$$

and substituting into the energy expressions to obtain the energy of the lower edge half-segment with a guided-pinned boundary, the membrane energy becomes

$$(V_1)_{j=N}=\frac{h\mu\pi}{1-\nu}\Big\{\Big[\frac{\alpha^2}{4}(u_N^2-2u_Nu_{N-1}+u_{N-1}^2)+\alpha(1+\nu)(u_N^2$$

$$-u_{N-1}u_N)\tan\bar\theta+2(1+\nu)u_N^2\tan^2\bar\theta\Big]\Big[\sin\bar\theta\sin 1/\alpha \qquad (121)$$

$$-(1-\cos 1/\alpha)\cos\bar{\theta}\Big] + \Big[2(1+\nu)u_N^2 \tan\bar{\theta} + \nu\alpha(u_N^2$$

$$-u_N u_{N-1})\Big]\Big[\cos\bar{\theta}\sin 1/\alpha + (1-\cos 1/\alpha)\sin\bar{\theta}\Big]$$

$$+u_N^2\left[(1-\cos 1/\alpha)\cos\bar{\theta}-\sin\bar{\theta}\sin 1/\alpha+\ln\tan\frac{\bar{\theta}}{2}-\ln\tan\frac{2N-1}{2\alpha}\right]\Bigg\}$$

Using (40) in (117) gives

$$(\Omega)_{j=N}=-\frac{\pi\rho h a^2}{2}\,\omega^2(1+\cot^2\bar{\theta})\left[\sin\bar{\theta}\sin 1/\alpha-(1-\cos 1/\alpha)\cos\bar{\theta}\right]u_N^2. \tag{122}$$

To obtain the single eq. of motion for the lower edge node, one considers only

$$\frac{\partial V}{\partial u_N}=0 \tag{123}$$

There is only one eq. of motion for the N-node since u_N + w_N are not independent as shown by (40). Thus, one finds

$$(V_N)=(U_1)_{j=N}+(U_2)_{j=N}+(\Omega)_{j=N}+(U_1)_{j=N-1}$$
$$+(U_2)_{j=N-1} \tag{124}$$

Thus using (124) in (123) the eq. of motion for the lower edge node is given by

$$\frac{\partial(U_1)_{j=N}}{\partial u_N}+\frac{\partial(U_1)_{j=N-1}}{\partial u_N}+\frac{\partial(U_2)_{j=N}}{\partial u_N}+\frac{\partial(U_2)_{j=N-1}}{\partial u_N}$$
$$+\frac{\partial(\Omega)_{j=N}}{\partial u_N}=0\ , \tag{125}$$

Substitution of the various energies into (11.) yields the equation of motion for the lower edge node of a guided-pinned boundary as follows:

$$(S^{uu}_{N\,N-1})u_{N-1}+(S^{uu}_{N\,N})u_N+(S^{uw}_{N\,N-2})w_{N-2}$$
$$+(S^{uw}_{N\,N-1})w_{N-1}-(M^{uu}_{N\,N})u_N=0\ , \tag{126}$$

where

$$S^{uu}_{N\,N-1} = \frac{h\mu\pi}{1-\nu}\Big\{\Big[\frac{\alpha^2}{2}+\alpha(1+\nu)\tan\bar{\theta}\Big]\Big[(1-\cos 1/\alpha)\cos\bar{\theta}-\sin\bar{\theta}\sin 1/\alpha\Big]$$

$$-\alpha\nu\Big[\cos\bar{\theta}\sin 1/\alpha+(1-\cos 1/\alpha)\sin\bar{\theta}\Big]-\alpha^2\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha$$

$$+2\alpha\nu\cos\frac{2(N-1)}{\alpha}\sin 1/\alpha\Big\},$$

$$S^{uu}_{N\,N} = \frac{h\mu\pi}{1-\nu}\Big\{\Big[\frac{\alpha^2}{2}+2(1+\nu)(\tan\bar{\theta})(\alpha+2\tan\bar{\theta})\Big]\Big[\sin\bar{\theta}\sin 1/\alpha$$

$$-(1-\cos 1/\alpha)\cos\bar{\theta}\Big]+2\Big[\alpha\nu+2(1+\nu)\tan\bar{\theta}\Big]\Big[\cos\bar{\theta}\sin 1/\alpha$$

$$+(1-\cos 1/\alpha)\sin\bar{\theta}\Big]+2\Big[(1-\cos 1/\alpha)\cos\bar{\theta}-\sin\bar{\theta}\sin 1/\alpha$$

$$+\ln\tan\frac{\bar{\theta}}{2}-\ln\tan\frac{2N-1}{2\alpha}\Big]+\alpha^2\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha\Big\}$$

$$+\frac{\mu\pi h^3}{12(1-\nu)\alpha^2}\Big\{\frac{\alpha^2}{2}\tan^2\bar{\theta}\Big[(\alpha-\nu\cot\bar{\theta})^2+2\alpha(\alpha-\nu\cot\bar{\theta})$$

$$+\alpha^2\Big]\Big[\sin\bar{\theta}\sin 1/\alpha-(1-\cos 1/\alpha)\cos\bar{\theta}\Big]+\nu^2\alpha^2\tan\bar{\theta}\Big[\alpha-\nu\cot\bar{\theta}$$

$$-\alpha\Big]\Big[\cos\bar{\theta}\sin 1/\alpha+(1-\cos 1/\alpha)\sin\bar{\theta}\Big]+\frac{\alpha^2}{2}\tan^2\bar{\theta}\Big[(1$$

$$-\cos 1/\alpha)\cos\bar{\theta}-\sin\bar{\theta}\sin 1/\alpha+\ln\tan\frac{\bar{\theta}}{2}-\ln\tan\frac{2N-1}{2\alpha}\Big]$$

$$+\Big[\frac{\alpha^4}{4}\tan^2\bar{\theta}\Big]\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha+\Big[\nu^2\alpha^3\tan^2\bar{\theta}\Big]\cos\frac{2(N-1)}{\alpha}\sin 1/\alpha$$

$$+\frac{\alpha^2}{2}\tan^2\bar{\theta}\Big[2\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha+\ln\tan\frac{2N-1}{2\alpha}-\ln\tan\frac{2N-3}{2\alpha}\Big]$$

$$S^{uw}_{N\,N-2} = \frac{\mu\pi h^3}{12(1-\nu)\alpha^2}\Big\{\Big[\frac{\alpha^4}{4}\tan\bar{\theta}\Big]\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha$$

$$+\Big[\frac{\nu^2\alpha^3}{2}\tan\bar{\theta}\Big]\cos\frac{2(N-1)}{\alpha}\sin 1/\alpha\Big\},$$

$$S^{uw}_{N\,N-1} = \frac{h\mu\pi}{1-\nu}\Big\{\Big[2\alpha(1+\nu)\Big]\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha\Big\}$$

$$+\frac{\mu\pi h^3}{12(1-\nu)\alpha^2}\Big\{\frac{\alpha^2}{4}\tan\bar{\theta}\Big[(\alpha-\nu\cot\bar{\theta})(\alpha-2\nu\cot\bar{\theta})$$

$$-\alpha(\alpha-2\nu\cot\bar{\theta})-\alpha(\alpha-\nu\cot\bar{\theta})+\alpha^2\Big]\Big[(1$$

$$-\cos 1/\alpha)\cos\bar{\theta}-\sin\bar{\theta}\sin 1/\alpha\Big]-\frac{\nu^2}{4}^2\tan\bar{\theta}\Big[(\alpha-$$

$$2\nu\cot\bar{\theta})+2(\alpha-\nu\cot\bar{\theta})-3\alpha\Big]\Big[\cos\bar{\theta}\sin 1/\alpha$$

$$+(1-\cos 1/\alpha)\sin\bar{\theta}\Big]-\frac{\alpha}{2}^2\tan\bar{\theta}\Big[(1-\cos 1/\alpha)\cos\bar{\theta}$$

$$-\sin\bar{\theta}\sin 1/\alpha+\ln\tan\frac{\bar{\theta}}{2}-\ln\tan\frac{2N-1}{2\alpha}\Big]$$

$$-\Big[\frac{\alpha^4}{2}\tan\bar{\theta}\Big]\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha-\Big[\frac{3\nu^2\alpha^3}{\alpha}\tan\bar{\theta}\Big]\cos\frac{2(N-1)}{\alpha}\sin 1/\alpha$$

$$-\frac{\alpha}{2}^2\tan\bar{\theta}\Big[-2\sin\frac{\alpha(N-1)}{\alpha}\sin 1/\alpha+\ln\tan\frac{2N-1}{2\alpha}-\ln\tan\frac{2N-3}{2\alpha}\Big]\Big\}$$

and $$m^{uu}_{NN}=\pi\rho h\,\alpha^2\omega^3(1+\tan^2\bar{\theta})\Big[\sin\bar{\theta}\sin 1/\alpha-(1-\cos 1/\alpha)\cos\bar{\theta}\Big].$$

Equations of Motion for the j=N-1 Point

The eqs. of motion for the j=N-1 segment can be found from

$$\frac{\partial V}{\partial u_{N-1}} = 0 \ , \tag{127}$$

and

$$\frac{\partial V}{\partial w_{N-1}} = 0 \ . \tag{128}$$

For (127) one notes that

$$V(u_{N-1}) = (\mathcal{V}_1)_{j=N-2} + (\mathcal{V}_1)_{j=N-1} + (\mathcal{V}_1)_{j=N} + (\Omega)_{j=N-1} \tag{129}$$

and for (128),

$$V(w_{N-1}) = (\mathcal{V}_1)_{j=N-1} + (\mathcal{V}_2)_{j=N-2} + (\mathcal{V}_2)_{j=N-1} + (\mathcal{V}_2)_{j=N} + (\Omega)_{j=N-1} \tag{130}$$

Thus using (127) and (130) in (128)and(127) respectively,the eqs. of

motion become

$$\frac{\partial(\mathcal{V}_1)_{j=N-2}}{\partial u_{N-1}} + \frac{\partial(\mathcal{V}_1)_{j=N-1}}{\partial u_{N-1}} + \frac{\partial(\mathcal{U}_1)_{j=N}}{\partial u_N} + \frac{\partial(\Omega)_{j=N-1}}{\partial u_N} = 0 \tag{131}$$

and

$$\frac{\partial(\mathcal{V}_1)_{j=N-1}}{\partial w_{N-1}} + \frac{\partial(\mathcal{V}_2)_{j=N-2}}{\partial w_{N-1}} + \frac{\partial(\mathcal{V}_2)_{j=N-1}}{\partial w_{N-1}} + \frac{\partial(\mathcal{V}_2)_{j=N}}{\partial w_{N-1}} + \frac{\partial(\Omega)_{j=N-1}}{\partial w_{N-1}} = 0 \tag{132}$$

Substitution of the various energies yields the u_{N-1} equation of motion

as follows:

$$\begin{aligned}(S^{u\ \ u}_{N-1\ N-2})u_{N-2} + (S^{u\ \ u}_{N-1\ N-1})u_{N-1} + (S^{u\ \ u}_{N-1\ N})u_N \\ + (S^{u\ \ w}_{N-1\ N-2})w_{N-2} + (S^{u\ \ w}_{N-1\ N-1})w_{N-1} - (M^{u\ \ u}_{N-1\ N-1})\ddot{u}_{N-1} = 0,\end{aligned} \tag{133}$$

where

$$S^{u\,u}_{N-1\,N-2} = \frac{h\mu\pi}{1-\nu}\left[-\alpha^2 \sin\frac{2(N-2)}{\alpha}\sin 1/\alpha + 2\alpha\nu\cos\frac{2(N-2)}{\alpha}\sin 1/\alpha\right],$$

$$S^{u\,u}_{N-1\,N-1} = \frac{h\mu\pi}{1-\nu}\left\{\alpha^2\sin\frac{2(N-2)}{\alpha}\sin 1/\alpha + \alpha^2\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha\right.$$
$$+\frac{\alpha^2}{2}\left[\sin\bar{\theta}\sin 1/\alpha - (1-\cos 1/\alpha)\cos\bar{\theta}\right] - 4\alpha\nu\cos\frac{2(N-1)}{\alpha}\sin 1/\alpha$$
$$\left.+2\left[2\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha + \ln\tan\frac{2N-1}{2\alpha} - \ln\tan\frac{2N-3}{2\alpha}\right]\right\},$$

$$S^{u\,u}_{N-1\,N} = \frac{h\mu\pi}{1-\nu}\left\{-\alpha^2\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha + 2\alpha\nu\cos\frac{2(N-1)}{\alpha}\sin 1/\alpha\right.$$
$$+\left[\frac{\alpha^2}{2} + \alpha(1+\nu)\tan\bar{\theta}\right]\left[(1-\cos 1/\alpha)\cos\bar{\theta} - \sin\bar{\theta}\sin 1/\alpha\right]$$
$$\left.-\alpha\nu\left[\cos\bar{\theta}\sin 1/\alpha + (1-\cos 1/\alpha)\sin\bar{\theta}\right]\right\},$$

$$S^{u\,w}_{N-1\,N-2} = \frac{h\mu\pi}{1-\nu}\left[2\alpha(1+\nu)\sin\frac{2(N-2)}{\alpha}\sin 1/\alpha\right],$$

$$S^{u\,w}_{N-1\,N-1} = \frac{h\mu\pi}{1-\nu}\left[-2\alpha(1+\nu)\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha + 4(1+\nu)\cos\frac{2(N-1)}{\alpha}\sin 1/\alpha\right],$$

and

$$m^{u\,u}_{N-1\,N-1} = 2\pi\rho h a^2\omega^2\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha\,.$$

The w_{N-1} equation of motion is determined in a similar manner and is given by equation (134)

$$(S^{w\,u}_{N-1\,N-1})u_{N-1} + (S^{w\,u}_{N-1\,N})u_N + (S^{w\,w}_{N-1\,N-3})w_{N-3}$$
$$+(S^{w\,w}_{N-1\,N-2})w_{N-2} + (S^{w\,w}_{N-1\,N-1})w_{N-1} - (m^{w\,w}_{N-1\,N-1})w_{N-1} = 0, \quad (134)$$

where

$$S^{w\,u}_{N-1\,N-1} = \frac{h\mu\pi}{1-\nu}\left[-2\alpha(1+\nu)\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha + 4(1+\nu)\cos\frac{2(N-1)}{\alpha}\sin 1/\alpha\right]$$

$$S^{w\,u}_{N-1\,N} = \frac{h\mu\pi}{1-\nu}\left[2\alpha(1+\nu)\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha\right] + \frac{\mu\pi h^3}{12(1-\nu)a^2}\left\{\right.$$
$$-\frac{\alpha^4}{2}(\tan\bar{\theta})\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha - \frac{3\nu^2\alpha^3}{2}(\tan\bar{\theta})\cos\frac{2(N-1)}{\alpha}\sin 1/\alpha$$

$$+\frac{\alpha^2}{4}(\tan\bar{\theta})\Big[+(\alpha-\nu\cot\bar{\theta})(\alpha-2\nu\cot\bar{\theta})-\alpha(\alpha-2\nu\cot\bar{\theta})$$

$$-\alpha(\alpha-\nu\cot\bar{\theta})+\alpha^2\Big]\Big[(1-\cos 1/\alpha)\cos\bar{\theta}$$

$$-\sin\bar{\theta}\sin 1/\alpha\Big]-\frac{\nu^2\alpha^2}{4}(\tan\bar{\theta})\Big[+(\alpha-2\nu\cot\bar{\theta})+2(\alpha-\nu\cot\bar{\theta})$$

$$-3\alpha\Big]\Big[\cos\bar{\theta}\sin 1/\alpha+(1-\cos 1/\alpha)\sin\bar{\theta}\Big]-\frac{\alpha^2}{2}\tan\bar{\theta}\Big[(1$$

$$-\cos 1/\alpha)\cos\bar{\theta}-\sin\bar{\theta}\sin 1/\alpha+\ln\tan\bar{\theta}/2$$

$$-\ln\tan\frac{2N-1}{2\alpha}\Big]-\frac{\alpha^2}{2}\tan\bar{\theta}\Big[-2\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha$$

$$+\ln\tan\frac{2N-1}{2\alpha}-\ln\tan\frac{2N-3}{2\alpha}\Big]\Big\}\quad,$$

$$S^{ww}_{N-1\;N-1}=\frac{h\mu\pi}{1-\nu}\Big[8(1+\nu)\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha\Big]+\frac{\mu\pi h^3}{12(1-\nu)a^2}$$

$$\Big\{\frac{\alpha^4}{4}\sin\frac{2(N-2)}{\alpha}\sin 1/\alpha+\alpha^4\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha+\frac{\alpha^2}{8}\Big[(\alpha$$

$$-2\nu\cot\bar{\theta})^2-2\alpha(\alpha-2\nu\cot\bar{\theta})\Big]\Big[\sin\bar{\theta}\sin 1/\alpha-(1$$

$$-\cos 1/\alpha)\cos\bar{\theta}\Big]+\nu^2\alpha^3\cos\frac{2(N-2)}{\alpha}\sin 1/\alpha+2\nu^2\alpha^3\cos\times$$

$$\frac{2(N-1)}{\alpha}\sin 1/\alpha+\frac{\nu^2\alpha^2}{2}\Big[(\alpha-2\nu\cot\bar{\theta})-\alpha\Big]\Big[\cos\bar{\theta}\sin 1/\alpha$$

$$+(1-\cos 1/\alpha)\sin\bar{\theta}\Big]+\frac{\alpha^2}{2}\Big[\circ\sin\frac{2(N-2)}{\alpha}\sin 1/\alpha+\ln\tan\frac{2N-3}{2\alpha}$$

$$-\ln\tan\frac{2N-5}{2\alpha}\Big]+\frac{\alpha^2}{2}(1.000)\Big[-2\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha$$

$$+\ln\tan\frac{2N-1}{2\alpha}-\ln\tan\frac{2N-3}{2\alpha}\Big]+\frac{\alpha^2}{2}\Big[(1-\cos 1/\alpha)\times$$

$$\cos\bar{\theta}-\sin\bar{\theta}\sin 1/\alpha+\ln\tan\bar{\theta}/2-\ln\tan\frac{2N-1}{2\alpha}\Big]\Big\}$$

and

$$m^{ww}_{N-1\;N-1}=2\pi\rho h a^2\omega^2\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha\quad,$$

$$S^{w\,w}_{N-1\;N-3} = \frac{\mu\pi h^3}{12(1-\nu)a^2}\;\frac{\alpha^4}{4}\sin\frac{2(N-2)}{\alpha}\sin 1/\alpha + \frac{\nu^2\alpha^3}{2}\cos\frac{2(N-2)}{\alpha}\times$$

$$\sin 1/\alpha$$

$$S^{w\,w}_{N-1\;N-2} = -\frac{\mu\pi h^3}{12(1-\nu)a^2}\;\frac{\alpha^4}{2}\sin\frac{2(N-2)}{\alpha}\sin 1/\alpha + \frac{\alpha^4}{2}\sin\frac{2(N-1)}{\alpha}\times$$

$$\sin 1/\alpha + \frac{3\nu^2\alpha^3}{2}\cos\frac{2(N-2)}{\alpha}\sin 1/\alpha + \frac{\nu^2\alpha^3}{2}\cos\times$$

$$\frac{2(N-1)}{\alpha}\sin 1/\alpha + \frac{\alpha^2}{2}\left[2\cos\frac{2(N-2)}{\alpha}\cos 1/\alpha\right.$$

$$\left. + \ln\tan\frac{2N-3}{2\alpha} - \ln\tan\frac{2N-5}{2\alpha}\right] .$$

Equations of Motion for the j=N-2 Point

At first, it appears that the 2nd point off the boundary of the shell would be a general point. However, as in the case of the dome point, this is not true of the j=N-2 node. The eqs. of motion of the lower edge node except for the coefficient $S^{u\ w}_{N\ N-2}$, since all coefficients except this one check out as symmetric, are given by the general eqs. of motion, i.e. eqs. (108). Thus the only eq. to be formulated in this section is

$$\frac{\partial V}{\partial w_{N-2}} = 0 \tag{135}$$

For (135) one can note that

$$V(w_{N-2}) = (U_{1,1})_{j=N-2} + (U_{1,2})_{j=N-3} + (U_{,2})_{j=N-2} + (U_2)_{j=N-1} + (\Omega)_{j=N-2} \tag{136}$$

Thus using (136)in (137)gives the equation of motion as

$$\frac{\partial(U_1)_{j=N-2}}{\partial w_{N-2}} + \frac{\partial(U_2)_{j=N-3}}{\partial w_{N-2}} + \frac{\partial(U_2)_{j=N-2}}{\partial w_{N-2}} + \frac{\partial(U_2)_{j=N-1}}{\partial w_{N-1}} + \frac{\partial(\Omega)_{j=N-2}}{\partial w_{N-2}} = 0 \tag{137}$$

Evaluation of the various coefficients defined yields,

$$\begin{aligned}&(S^{w\ u}_{N-2\ N-2})u_{N-2} + (S^{w\ u}_{N-2\ N-1})u_{N-1} + (S^{w\ u}_{N-2\ N})u_N + (S^{w\ w}_{N-2\ N-4})\\ &\times w_{N-4} + (S^{w\ w}_{N-2\ N-3})w_{N-3} + (S^{w\ w}_{N-2\ N-2})w_{N-2} + (S^{w\ w}_{N-2\ N-1})w_{N-1}\\ &- (M^{w\ w}_{N-2\ N-2})w_{N-2} = 0\end{aligned} \tag{138}$$

where

$$S^{w\ u}_{N-2\ N-2} = \frac{h\mu\pi}{1-\nu}\left[-2\alpha(1+\nu)\sin\frac{2(N-2)}{\alpha}\cos 1/\alpha + 4(1+\nu)\cos\frac{2(N-2)}{\alpha} \times \sin 1/\alpha\right],$$

$$S^{w\ u}_{N-2\ N-1} = \frac{h\mu\pi}{1-\nu}\left[2\alpha(1+\nu)\sin\frac{2(N-2)}{\alpha}\cos 1/\alpha\right],$$

$$S^{w\ u}_{N-2\ N} = \frac{\mu \pi h^3}{12(1-\nu)a^2}\left[\frac{\alpha^4}{4}(\tan\bar{\theta})\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha\right.$$

$$\left. + \frac{\nu^2\alpha^3}{2}(\tan\bar{\theta})\cos\frac{2(N-1)}{\alpha}\sin 1/\alpha\right.$$

$$S^{w\ w}_{N-2\ N-4} = \frac{\mu \pi h^3}{12(1-\nu)a^2}\left[\frac{\alpha^4}{4}\sin\frac{2(N-3)}{\alpha}\sin 1/\alpha + \frac{\nu^2\alpha^3}{2}\cos\frac{2(N-3)}{\alpha}\sin 1/\alpha\right],$$

$$S^{w\ w}_{N-2\ N-3} = \frac{-\mu \pi h^3}{12(1-\nu)a^2}\left\{\frac{\alpha^4}{2}\sin\frac{2(N-3)}{\alpha}\sin 1/\alpha + \frac{\alpha^4}{2}\sin\frac{2(N-2)}{\alpha}\sin 1/\alpha\right.$$

$$+ \frac{3\nu^2\alpha^3}{2}\cos\frac{2(N-3)}{\alpha}\sin 1/\alpha + \frac{\nu^2\alpha^3}{2}\cos\frac{2(N-2)}{\alpha}\sin 1/\alpha$$

$$\left. + \frac{\alpha^2}{2}\left[2\cos\frac{2(N-3)}{\alpha}\cos 1/\alpha + \ln\tan\frac{2N-5}{2\alpha} - \ln\tan\frac{2N-2}{2\alpha}\right]\right\}$$

$$S^{w\ w}_{N-2\ N-2} = \frac{h\mu\pi}{1-\nu}\left[8(1+\nu)\sin\frac{2(N-2)}{\alpha}\cos 1/\alpha\right] + \frac{\mu\pi h^3}{12(1-\nu)a^2}$$

$$\times\left\{\frac{\alpha^4}{4}\sin\frac{2(N-2)}{\alpha}\sin 1/\alpha + \alpha^4\sin\frac{2(N-2)}{\alpha}\sin 1/\alpha\right.$$

$$+ \frac{\alpha^4}{4}\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha + \nu^2\alpha^3\cos\frac{2(N-3)}{\alpha}\sin 1/\alpha$$

$$+ 2\nu\alpha^3\cos\frac{2(N-2)}{\alpha}\sin 1/\alpha + \frac{\alpha^2}{2}\left[2\cos\frac{2(N-5)}{\alpha}\cos 1/\alpha\right.$$

$$\left. + \ln\tan\frac{2N-5}{2\alpha} - \ln\tan\frac{2N-7}{2\alpha}\right] + \frac{\alpha^2}{\alpha}\left[2\cos\frac{2(N-2)}{\alpha}\cos 1/\alpha\right.$$

$$\left.\left. + \ln\tan\frac{2N-3}{2\alpha} - \ln\tan\frac{2N-5}{2\alpha}\right]\right\}\quad ,$$

$$S^{w\ w}_{N-2\ N-1} = -\frac{\mu\pi h^3}{12(1-\nu)a^2}\left\{\frac{\alpha^4}{2}\sin\frac{2(N-2)}{\alpha}\sin 1/\alpha + \frac{\alpha^4}{2}\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha\right.$$

$$+ \frac{3\nu^2\alpha^3}{2}\cos\frac{2(N-2)}{\alpha}\sin 1/\alpha + \frac{\nu^2\alpha^3}{2}\cos\frac{2(N-1)}{\alpha}\sin 1/\alpha$$

$$\left. + \frac{\alpha^2}{2}\left[2\cos\frac{2(N-2)}{\alpha}\cos 1/\alpha + \ln\tan\frac{2N-3}{2\alpha} - \ln\tan\frac{2N-5}{2\alpha}\right]\right\}$$

and

$$M^{w\ w}_{N-2\ N-2} = 2\pi\rho h a^2 w^2 \sin\frac{2(N-2)}{\alpha}\sin 1/\alpha$$

PROBLEM FORMULATION

Summary of Equations of Motion

Since all the required equations of motion have now been derived it is possible to gather all the eqs. of motion for the guided-pinned spherical cap for easier reference. They are as follows:

j=0: eq. of motion w.r.t. w_0 (dome point):

$$(S^{wu}_{01})u_1+(S^{ww}_{00})w_0+(S^{ww}_{01})w_1+(S^{ww}_{02})w_2-(M^{ww}_{00})w_0=0, \tag{139}$$

j=1: eq. of motion w.r.t. u_1:

$$(S^{uu}_{11})u_1+(S^{uu}_{12})u_2+(S^{uw}_{10})w_0+(S^{uw}_{11})w_1-(M^{uu}_{11})u_1=0 \tag{140}$$

eqs. of motion w.r.t. w_1:

$$(S^{wu}_{11})u_1+(S^{wu}_{12})u_2+(S^{ww}_{10})w_0+(S^{ww}_{11})w_1+(S^{ww}_{12})w_2$$
$$+(S^{ww}_{13})w_3-(M^{ww}_{11})w_1=0, \tag{141}$$

j=2: eq. of motion w.r.t. u_2(with k=2):

$$(S^{uu}_{21})u_1+(S^{uu}_{22})u_2+(S^{uu}_{23})u_3+(S^{uw}_{21})w_1+(S^{uw}_{22})w_2$$
$$-(M^{uu}_{22})u_2=0, \tag{142}$$

eqs. of motion w.r.t. w_2:

$$(S^{wu}_{22})u_2+(S^{wu}_{23})u_3+(S^{ww}_{20})w_0+(S^{ww}_{21})w_1 \tag{143}$$
$$+(S^{ww}_{22})w_2+(S^{ww}_{23})w_3+(S^{ww}_{24})w_4-(M^{ww}_{22})w_2=0,$$

j=k: (k = 3,4,...,N-4, N-3):

eqs. of motion w.r.t. u_k

$$(S^{uu}_{k\,k-1})u_{k-1}+(S^{uu}_{kk})u_k+(S^{uu}_{k\,k+1})u_{k+1}+(S^{uw}_{k\,k-1})w_{k-1}$$
$$+(S^{uw}_{kk})w_k-(M^{uu}_{kk})u_k=0, \tag{144}$$

eqs. of motion w.r.t. w_k:

$$(S^{wu}_{kk})u_k+(S^{wu}_{k\,k+1})u_{k+1}+(S^{ww}_{k\,k-2})w_{k-2}+(S^{ww}_{k\,k-1})w_{k-1}$$
$$+(S^{ww}_{kk})w_k+(S^{ww}_{k\,k+1})w_{k+1}+(S^{ww}_{k\,k+2})w_{k+2}-(M^{ww}_{kk})w_k=0, \tag{145}$$

j=N-2:

eqs. of motion w.r.t. u_{N-2} with k=N-2:

$$(S^{u}_{k-2}{}^{u}_{N-3})u_{N-3}+(S^{u}_{N-2}{}^{u}_{N-2})u_{N-2}+(S^{u}_{N-2}{}^{u}_{N-1})u_{N-1}+(S^{u}_{N-2}{}^{w}_{N-3})\times$$
$$w_{N-3}+(S^{u}_{N-2}{}^{w}_{N-2})w_{N-2}-(m^{u}_{N-2}{}^{u}_{N-2})u_{N-2}=0\ , \qquad (146)$$

eqs. of motion w.r.t. w_{N-2}:

$$(S^{w}_{N-2}{}^{u}_{N-2})u_{N-2}+(S^{w}_{N-2}{}^{u}_{N-1})u_{N-1}+(S^{w}_{N-2}{}^{u}_{N})u_{N}+(S^{w}_{N-2}{}^{w}_{N-2})w_{N-4}$$
$$+(S^{w}_{N-2}{}^{w}_{N-3})w_{N-3}+(S^{w}_{N-2}{}^{w}_{N-2})w_{N-2}+(S^{w}_{N-2}{}^{w}_{N-1})w_{N-1}-(m^{w}_{N-2}{}^{w}_{N-2})w_{N-2}=0, \qquad (147)$$

j=N-1:

eqs. of motion w.r.t. u_{N-1}:

$$(S^{u}_{N-1}{}^{u}_{N-2})u_{N-2}+(S^{u}_{N-1}{}^{u}_{N-1})u_{N-1}+(S^{u}_{N-1}{}^{u}_{N})u_{N}+(S^{u}_{N-1}{}^{w}_{N-2})w_{N-2}$$
$$+(S^{u}_{N-1}{}^{w}_{N-1})w_{N-1}-(m^{u}_{N-1}{}^{u}_{N-1})u_{N-1}=0, \qquad (148)$$

eqs. of motion w.r.t. w_{N-1}:

$$(S^{w}_{N-1}{}^{u}_{N-1})u_{N-1}+(S^{w}_{N-1}{}^{u}_{N})u_{N}+(S^{w}_{N-1}{}^{w}_{N-3})w_{N-3}+(S^{w}_{N-1}{}^{w}_{N-2})w_{N-2}$$
$$+(S^{w}_{N-1}{}^{w}_{N-1})w_{N-1}-(m^{w}_{N-1}{}^{w}_{N-1})w_{N-1}=0, \qquad (149)$$

j=N (lower edge boundary node):

eqs. of motion w.r.t. u_N:

$$(S^{u}_{N}{}^{u}_{N-1})u_{N-1}+(S^{u}_{N}{}^{u}_{N})u_{N}+(S^{u}_{N}{}^{w}_{N-2})w_{N-2}$$
$$+(S^{u}_{N}{}^{w}_{N-1})w_{N-1}-(m^{u}_{N}{}^{u}_{N})u_{N}=0 \ . \qquad (150)$$

Matrix Representation

It is evident that most of the stiffness coefficients have a symmetric property. Hence the eqs. of motion can be arranged in such an order that they will have a symmetric stiffness matrix. From McDonald's paper [9] one observes that there appears to be two methods of arrangement for the displacement vector. They are for our case

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \\ w_0 \\ w_1 \\ \vdots \\ w_{N-2} \\ w_{N-1} \end{bmatrix} \quad \text{AND} \quad \begin{bmatrix} w_0 \\ u_1 \\ w_1 \\ u_2 \\ w_2 \\ \vdots \\ u_{N-2} \\ w_{N-2} \\ u_{N-1} \\ w_{N-1} \\ u_N \end{bmatrix} .$$

The first type looks the most promising since to obtain the second type vector, one would have to put the 3 eqs. of $j=0$ and $j=1$ in some symmetrical form and this seems to be impossible. Therefore, consider all the u eqs. and then all the w equations. Hence, one obtains the following stiffness matrix:

$$
\left[\begin{array}{c|c}
\begin{matrix}
S^{uu}_{1\,1} & S^{uu}_{1\,2} & & & & & & & & & \\
& S^{uu}_{2\,2} & S^{uu}_{2\,3} & & & & & & & & \\
& & \ddots & \ddots & & & & & & & \\
& & & S^{uu}_{K-1\,K-1} & S^{uu}_{K-1\,K} & & & & & & \\
& & & & S^{uu}_{K\,K} & S^{uu}_{K\,K+1} & & & & & \\
& & & & & S^{uu}_{K+1\,K+1} & S^{uu}_{K+1\,K+2} & & & & \\
& & & & & & \ddots & \ddots & & & \\
& & & & & & & S^{uu}_{N-2\,N-2} & S^{uu}_{N-2\,N-1} & & \\
& & & & & & & & S^{uu}_{N-1\,N-1} & S^{uu}_{N-1\,N} & \\
& & & & & & & & & S^{uu}_{N\,N} & 0
\end{matrix}
&
\begin{matrix}
S^{uw}_{1\,0} & S^{uw}_{1\,1} & & & & & & & \\
& S^{uw}_{2\,1} & S^{uw}_{2\,2} & & & & & & \\
& & S^{uw}_{K-1\,K-2} & S^{uw}_{K-1\,K-1} & & & & & \\
& & & S^{uw}_{K\,K-1} & S^{uw}_{K\,K} & & & & \\
& & & & S^{uw}_{K+1\,K} & S^{uw}_{K+1\,K+1} & & & \\
& & & & & & S^{uw}_{N-2\,N-2} & & \\
& & & & & & S^{uw}_{N-1\,N-2} & S^{uw}_{N-1\,N-1} & \\
& & & & & & S^{uw}_{N\,N-2} & S^{uw}_{N\,N-1} &
\end{matrix}
\\ \hline
O
&
\begin{matrix}
S^{ww}_{0\,0} & S^{ww}_{0\,1} & S^{ww}_{0\,2} & & & & & & & & & \\
& S^{ww}_{1\,1} & S^{ww}_{1\,2} & S^{ww}_{1\,3} & & & & & & & & \\
& & S^{ww}_{2\,2} & S^{ww}_{2\,3} & S^{ww}_{2\,4} & & & & & & & \\
& & & \ddots & \ddots & \ddots & & & & & & \\
& & & & S^{ww}_{K-2\,K-2} & S^{ww}_{K-2\,K-1} & S^{ww}_{K-2\,K} & & & & & \\
& & & & & S^{ww}_{K-1\,K-1} & S^{ww}_{K-1\,K} & S^{ww}_{K-1\,K+1} & & & & \\
& & & & & & S^{ww}_{K\,K} & S^{ww}_{K\,K+1} & S^{ww}_{K\,K+2} & & & \\
& & & & & & & S^{ww}_{K+1\,K+1} & S^{ww}_{K+1\,K+2} & S^{ww}_{K+1\,K+3} & & \\
& & & & & & & & S^{ww}_{K+2\,K+2} & S^{ww}_{K+2\,K+3} & S^{ww}_{K+2\,K+4} & \\
& & & & & & & & & \ddots & \ddots & \\
& & & & & & & & & & S^{ww}_{N-2\,N-2} & S^{ww}_{N-2\,N-1} \\
& & & & & & & & & & & S^{ww}_{N-1\,N-1}
\end{matrix}
\end{array}\right]
$$

The resulting mass matrix is a digonal matrix so that the notation can be shortened as follows: $m^{u\,u}_{1\,1} = m^{u}_{1}$, $m^{u\,u}_{2\,2} = m^{u}_{2}$, etc. The matrix can be written as

$$
\begin{bmatrix}
m^u_1 & & & & & & & & & & & & \\
& m^u_2 & & & & & & & & & 0 & & \\
& & \ddots & & & & & & & & & & \\
& & & m^u_k & & & & & & & & & \\
& & & & \ddots & & & & & & & & \\
& & & & & m^u_{N-1} & & & & & & & \\
& & & & & & m^u_N & & & & & & \\
& & & & & & & m^w_0 & & & & & \\
& & & & & & & & m^w_1 & & & & \\
& & & & & & & & & \ddots & & & \\
& & 0 & & & & & & & & m^w_k & & \\
& & & & & & & & & & & m^w_{N-2} & \\
& & & & & & & & & & & & m^w_{N-1}
\end{bmatrix}
\begin{bmatrix}
u_1 \\ u_2 \\ \vdots \\ u_k \\ \vdots \\ u_{N-1} \\ u_N \\ w_0 \\ w_1 \\ \vdots \\ w_k \\ \vdots \\ w_{N-2} \\ w_{N-1}
\end{bmatrix}
$$

Therefore, the system of 2(N+1)-2 eqs. of motion can be written as

$$\{[S]-[m]\}\ [q] = 0 \quad . \tag{151}$$

Now, one can group into dimensionless quantities and rearrange terms in eqs.(151)by multiplying them by

$$\frac{(1-\nu)}{\mu\pi h} ,$$

so that the group coefficients of the energy expression are as follows:

$$U_1 : \frac{\mu\pi h}{1-\nu}\{\cdots\} \Rightarrow \{\cdots\cdots\} , \tag{152}$$

$$U_2 : \frac{\mu\pi h^3}{12(1-\nu)a^2}\{\cdots\} \Rightarrow \frac{1}{12}\left(\frac{h}{a}\right)^2\{\cdots\cdots\} .$$

Hence eqs. (1) can be written as

$$\{[C]-\lambda^2[m]\}\ [q] = 0 , \tag{153}$$

where [c] is now the stiffness matrix,

and
$$\lambda^2 = \left(\frac{\rho a^2}{E}\right)\omega^2\, 2(1+\nu)(1-\nu) , \tag{154}$$

where

$$\mu = E/2(1+\nu) .$$

Therefore the new mass matrix is of the form:

$$
\begin{bmatrix}
2\sin\frac{2}{\alpha}\sin\frac{1}{\alpha} & & & & & & & & & & & & \\
& 2\sin\frac{4}{\alpha}\sin\frac{1}{\alpha} & & & & & & & & & & 0 & \\
& & \ddots & & & & & & & & & & \\
& & & 2\sin\frac{2k}{\alpha}\sin\frac{1}{\alpha} & & & & & & & & & \\
& & & & \ddots & & & & & & & & \\
& & & & & 2\sin\frac{2(N-1)}{\alpha}\sin\frac{1}{\alpha} & & & & & & & \\
& & & & & & (1+\tan^2\bar{\theta})\left[\sin\bar{\theta}\sin\frac{1}{\alpha}-\left(1-\cos\frac{1}{\alpha}\right)\cos\bar{\theta}\right] & & & & & & \\
& & & & & & & \left(1-\cos\frac{1}{\alpha}\right) & & & & & \\
& & & & & & & & 2\sin\frac{2}{\alpha}\sin\frac{1}{\alpha} & & & & \\
& & & & & & & & & \ddots & & & \\
& 0 & & & & & & & & & 2\sin\frac{2k}{\alpha}\sin\frac{1}{\alpha} & & \\
& & & & & & & & & & & 2\sin\frac{2(N-2)}{\alpha}\sin\frac{1}{\alpha} & \\
& & & & & & & & & & & & 2\sin\frac{2(N-1)}{\alpha}\sin\frac{1}{\alpha}
\end{bmatrix}
\begin{bmatrix}
u_1 \\ u_2 \\ \vdots \\ u_k \\ \vdots \\ u_{N-1} \\ u_N \\ 2J_0 \\ w_1 \\ \vdots \\ w_k \\ w_{N-2} \\ w_{N-1}
\end{bmatrix}
\qquad (155)
$$

Now define

$$\text{Coef} = \frac{\lambda^2}{2(1-\nu^2)} .$$

Then

$$\omega^2 = \text{Coef} \left[\frac{E}{\rho a^2} \right] . \qquad (156)$$

Now the stiffness coefficients of the stiffness matrix from the eqs. of motion will be redefined to coincide with eqs. (153) Thus the coefficients are listed as follows:

$$C^{uu}_{11} = \alpha^2 \left[\frac{1-\cos 1/\alpha}{2} + \sin 2/\alpha \sin 1/\alpha \right] - 4\alpha\nu\cos 2/\alpha \sin 1/\alpha$$
$$+ 2\left[-2\sin 2/\alpha \sin 1/\alpha + \ln\tan 3/\alpha - \ln\tan 1/2\alpha \right], \qquad (157)$$

$$C^{uu}_{12} = -\alpha^2 \sin 2/\alpha \sin 1/\alpha - 2\alpha\nu\, 2/\alpha \sin 1/\alpha , \qquad (158)$$

$$C^{uw}_{10} = \alpha(1+\nu)(1-\cos 1/\alpha), \qquad (159)$$

$$C^{uw}_{11} = -2\alpha(1+\nu)\sin 2/\alpha \sin 1/\alpha + 4(1+\nu)\cos 2/\alpha \sin 1/\alpha , \qquad (160)$$

The coefficients of the u eqs. are given by the general expressions for k=2,3..., N-3, N-2, so that

$$C^{uu}_{kk} = \alpha^2 \left[\sin\frac{2k}{\alpha} + \sin\frac{2(k-1)}{\alpha} \right] \sin 1/\alpha$$
$$- 4\alpha\nu\cos\frac{2k}{\alpha} \sin 1/\alpha + 2\left[-2\sin\frac{2k}{\alpha} \sin 1/\alpha + \ln\tan\frac{2k+1}{2\alpha} \right.$$
$$\left. - \ln\tan\frac{2k-1}{2\alpha} \right], \qquad (161)$$

$$C^{uu}_{k\,k+1} = -\alpha^2 \sin\frac{2k}{\alpha} \sin 1/\alpha + 2\alpha\nu\cos\frac{2k}{\alpha} \sin 1/\alpha , \qquad (162)$$

$$C^{uw}_{k\,k-1} = 2\alpha(1+\nu)\sin\frac{2(k-1)}{\alpha} \sin 1/\alpha , \qquad (163)$$

$$C^{uw}_{kk} = -2\alpha(1+\nu)\sin\frac{2k}{\alpha}\sin 1/\alpha + 4(1+\nu)\cos\frac{2k}{\alpha}\sin 1/\alpha, \quad (164)$$

and for N-1 and N

$$C^{u\ u}_{N-1\ N-1} = \alpha^2\left[\sin\frac{2(N-2)}{\alpha} + \sin\frac{2(N-1)}{\alpha}\right]\sin 1/\alpha + \frac{\alpha^2}{2}\left[\sin\bar\theta\sin 1/\alpha - (1-\cos 1/\alpha)\cos\bar\theta\right] - 4\alpha\nu\cos\frac{2(N-1)}{\alpha}\sin 1/\alpha + 2\left[-2\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha + \ln\tan\frac{2N-1}{2\alpha} - \ln\tan\frac{2N-3}{2\alpha}\right] \quad (165)$$

$$C^{u\ u}_{N-1\ N} = -\alpha^2\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha + 2\alpha\nu\cos\frac{2(N-1)}{\alpha}\sin 1/\alpha + \left[\frac{\alpha^2}{2} + \alpha(1+\nu)\tan\bar\theta\right]\left[(1-\cos 1/\alpha)\cos\bar\theta - \sin\bar\theta\sin 1/\alpha\right] - \alpha\nu\left[\cos\bar\theta\sin 1/\alpha + (1-\cos 1/\alpha)\sin\bar\theta\right], \quad (166)$$

$$C^{u\ w}_{N-1\ N-2} = 2\alpha(1+\nu)\sin\frac{2(N-2)}{\alpha}\sin 1/\alpha \quad , \quad (167)$$

$$C^{u\ w}_{N-1\ N-1} = -2\alpha(1+\nu)\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha + 4(1+\nu)\cos\frac{2(N-1)}{\alpha}\sin 1/\alpha; \quad (168)$$

$$C^{u\ u}_{N\ N} = \alpha^2\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha + \left[\frac{\alpha^2}{2} + 2(1+\nu)(\alpha + 2\tan\bar\theta)\tan\bar\theta\right] \quad (169)$$
$$\left[\sin\bar\theta\sin 1/\alpha - (1-\cos 1/\alpha)\cos\bar\theta\right] + 2\left[\alpha\nu + 2(1+\nu)\tan\bar\theta\right]$$
$$\left[\cos\bar\theta\sin 1/\alpha + (1-\cos 1/\alpha)\sin\bar\theta\right] + 2\left[(1-\cos 1/\alpha)\cos\bar\theta - \sin\bar\theta\sin 1/\alpha + \ln\tan\frac{\bar\theta}{2} - \ln\tan\frac{2N-1}{2\alpha}\right]$$
$$+ \frac{1}{12}\left(\frac{h}{a}\right)^2\left\{\frac{\alpha^2}{2}\tan^2\bar\theta\left[(\alpha - \nu\cot\bar\theta)^2 + 2\alpha(\alpha - \nu\cot\bar\theta) + \alpha^2\right]\left[\sin\bar\theta\sin 1/\alpha - (1-\cos 1/\alpha)\cos\bar\theta\right] - \nu^3\alpha^2\tan\bar\theta\right.$$
$$\left[\cos\bar\theta\sin 1/\alpha + (1-\cos 1/\alpha)\sin\bar\theta\right] + \frac{\alpha^2}{2}\tan^2\bar\theta\left[(1-\cos 1/\alpha)\cos\bar\theta\right.$$

$$-\sin\bar{\theta}\sin 1/\alpha+\ln\tan\frac{\bar{\theta}}{2}-\ln\tan\frac{2N-1}{2\alpha}\Big]$$

$$+\frac{\alpha^4}{4}\tan^2\bar{\theta}\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha+\nu^2\alpha^3\tan^2\bar{\theta}\cos\frac{2(N-1)}{\alpha}\sin 1/\alpha$$

$$+\frac{\alpha^2}{2}\tan^2\bar{\theta}\Big[-2\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha+\ln\tan\frac{2N-1}{2\alpha}-\ln\tan\frac{2N-3}{2\alpha}\Big]\Big\},$$

$$C^{u\,w}_{N\;N-2}=\frac{1}{12}\left(\frac{h}{a}\right)^2\left\{\frac{\alpha^4}{4}\sin\frac{2(N-1)}{\alpha}+\frac{\nu^2\alpha^3}{2}\cos\frac{2(N-1)}{\alpha}\right\}\sin 1/\alpha\,\tan\bar{\theta} \tag{170}$$

$$C^{u\,w}_{N\;N-1}=2\alpha(1+\nu)\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha+\frac{1}{12}\left(\frac{h}{a}\right)^2\Big\{\frac{\alpha^2}{4}\tan\bar{\theta}\Big[(\alpha$$

$$-\nu\cot\bar{\theta})(\alpha-2\nu\cot\bar{\theta})-\alpha(2\alpha-3\nu\cot\bar{\theta})+\alpha^2\Big]\Big[(1 \tag{171}$$

$$-\cos 1/\alpha)\cos\bar{\theta}-\sin\bar{\theta}\sin 1/\alpha\Big]+\nu^3\alpha^2(1.000)\Big[\cos\bar{\theta}\sin 1/\alpha$$

$$+(1-\cos 1/\alpha)\sin\bar{\theta}\Big]-\frac{\alpha^2}{2}\tan\bar{\theta}\Big[(1-\cos 1/\alpha)\cos\bar{\theta}-\sin\bar{\theta}\sin 1/\alpha$$

$$+\ln\tan\frac{\bar{\theta}}{2}-\ln\tan\frac{2N-1}{2\alpha}\Big]-\frac{\alpha^4}{2}\tan\bar{\theta}\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha$$

$$-\frac{3\nu^2\alpha^3}{2}\tan\bar{\theta}\cos\frac{2(N-1)}{\alpha}\sin 1/\alpha-\frac{\alpha^2}{2}\tan\bar{\theta}\Big[-2\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha$$

$$+\ln\tan\frac{2N-1}{2\alpha}-\ln\tan\frac{2N-3}{2\alpha}\Big]\Big\},$$

and the coefficients of the w eqs. are

$$C^{w\,w}_{0\;0}=4(1+\nu)(1-\cos 1/\alpha)+\frac{1}{12}\left(\frac{h}{a}\right)^2\Big\{\frac{\alpha^4}{8}\Big[(1-\cos 1/\alpha) \tag{172}$$

$$+2\sin 3/\alpha\,\sin 1/\alpha\Big]\Big\},$$

$$C^{w\,w}_{0\;1}=-\frac{1}{12}\left(\frac{h}{a}\right)^2\Big\{\frac{\alpha^4}{4}\Big[(1-\cos 1/\alpha)+2\sin 2/\alpha\,\sin 1/\alpha\Big]$$

$$+\frac{\nu^2\alpha^3}{2}\cos 2/\alpha\,\cos 1/\alpha\Big\}, \tag{173}$$

$$C^{w\,w}_{0\;2}=\frac{1}{12}\left(\frac{h}{a}\right)^2\Big\{\frac{\alpha^4}{8}\Big[(1-\cos 1/\alpha)+2\sin 2/\alpha\,\sin 1/\alpha\Big]$$

$$+\frac{\nu^2\alpha^3}{2}\cos 2/\alpha\,\sin 1/\alpha\Big\}, \tag{174}$$

$$C^{ww}_{11} = 8(1+\nu)\sin\tfrac{2}{\alpha}\sin\tfrac{1}{\alpha} + \frac{1}{12}\left(\frac{h}{a}\right)^2\Big\{\frac{\alpha^4}{4}\Big[2(1-\cos\tfrac{1}{\alpha}) + (4\sin\tfrac{2}{\alpha} + \sin\tfrac{4}{\alpha})\sin\tfrac{1}{\alpha}\Big] + 2\nu^2\alpha^3\cos\tfrac{2}{\alpha}\sin\tfrac{1}{\alpha} + \frac{\alpha^2}{2}\Big[-2\sin\tfrac{2}{\alpha}\sin\tfrac{1}{\alpha} + \ln\tan\tfrac{3}{2\alpha} - \ln\tan\tfrac{1}{2\alpha}\Big]\Big\} \tag{175}$$

$$C^{ww}_{12} = -\frac{1}{12}\left(\frac{h}{a}\right)^2\Big\{\frac{\alpha^4}{4}\Big[(1-\cos\tfrac{1}{\alpha}) + 2(\sin\tfrac{2}{\alpha} + \sin\tfrac{4}{\alpha})\sin\tfrac{1}{\alpha}\Big] + \frac{\nu^2\alpha^3}{2}\Big[3\cos\tfrac{2}{\alpha} + \cos\tfrac{4}{\alpha}\Big]\sin\tfrac{1}{\alpha} + \frac{\alpha^2}{2}\Big[-2\sin\tfrac{2}{\alpha}\sin\tfrac{1}{\alpha} + \ln\tan\tfrac{3}{2\alpha} - \ln\tan\tfrac{1}{2\alpha}\Big]\Big\}, \tag{176}$$

$$C^{ww}_{13} = \frac{1}{12}\left(\frac{h}{a}\right)^2\Big\{\frac{\alpha^4}{4}\sin\tfrac{4}{\alpha}\sin\tfrac{1}{\alpha} + \frac{\nu^2\alpha^3}{2}\cos\tfrac{4}{\alpha}\sin\tfrac{1}{\alpha}\Big\}, \tag{177}$$

$$C^{ww}_{22} = 2(1+\nu)\sin\tfrac{4}{\alpha}\sin\tfrac{1}{\alpha} + \frac{1}{12}\left(\frac{h}{a}\right)^2\Big\{\frac{\alpha^4}{8}\Big[(1-\cos\tfrac{1}{\alpha}) + 2(\sin\tfrac{2}{\alpha} + 4\sin\tfrac{4}{\alpha} + \sin\tfrac{6}{\alpha})\sin\tfrac{1}{\alpha}\Big] + \nu^2\alpha^3\Big[\cos\tfrac{2}{\alpha} + 2\cos\tfrac{4}{\alpha}\Big]\sin\tfrac{1}{\alpha} + \frac{\alpha^2}{2}\Big[-2(\sin\tfrac{2}{\alpha} + \sin\tfrac{4}{\alpha})\sin\tfrac{1}{\alpha} + \ln\tan\tfrac{5}{2\alpha} - \ln\tan\tfrac{1}{2\alpha}\Big]\Big\}, \tag{178}$$

$$C^{ww}_{23} = -\frac{1}{12}\left(\frac{h}{a}\right)^2\Big\{\frac{\alpha^4}{2}\Big[\sin\tfrac{4}{\alpha} + \sin\tfrac{6}{\alpha}\Big]\sin\tfrac{1}{\alpha} + \frac{\nu^2\alpha^3}{2}\Big[3\cos\tfrac{4}{\alpha} + \cos\tfrac{6}{\alpha}\Big]\sin\tfrac{1}{\alpha} + \frac{\alpha^2}{2}\Big[-2\sin\tfrac{4}{\alpha}\sin\tfrac{1}{\alpha} + \ln\tan\tfrac{5}{2\alpha} - \ln\tan\tfrac{3}{2\alpha}\Big]\Big\}, \tag{179}$$

$$C^{ww}_{24} = \frac{1}{12}\left(\frac{h}{a}\right)^2\Big[\frac{\alpha^4}{4}\sin\tfrac{6}{\alpha} + \frac{\nu^2\alpha^3}{2}\cos\tfrac{6}{\alpha}\Big]\sin\tfrac{1}{\alpha}, \tag{180}$$

and from general expressions where k=3, 4..., N-4, N-3

$$C^{ww}_{k\,k} = 8(1+\nu)\sin\frac{2k}{\alpha}\sin\frac{1}{\alpha} + \frac{1}{12}\left(\frac{h}{a}\right)^2\left\{\frac{\alpha^4}{4}\left[\sin\frac{2(k-1)}{\alpha} + 4\sin\frac{2k}{\alpha} + \sin\frac{2(k+1)}{\alpha}\right]\sin\frac{1}{\alpha} + \nu^2\alpha^3\left[\cos\frac{2(k-1)}{\alpha} + 2\cos\frac{2k}{\alpha}\right]\sin\frac{1}{\alpha} + \frac{\alpha^2}{2}\left[-2\left(\sin\frac{2(k-1)}{\alpha} + \sin\frac{2k}{\alpha}\right)\sin\frac{1}{\alpha} + \ln\tan\frac{2k+1}{2\alpha} - \ln\tan\frac{2k-3}{2\alpha}\right]\right\}, \tag{181}$$

$$C^{ww}_{k\,k+1} = -\frac{1}{12}\left(\frac{h}{a}\right)^2\left\{\frac{\alpha^4}{2}\left[\sin\frac{2k}{\alpha} + \sin\frac{2(k+1)}{\alpha}\right]\sin\frac{1}{\alpha} + \frac{\nu^2\alpha^3}{\alpha}\left[3\cos\frac{2k}{\alpha} + \cos\frac{2(k+1)}{\alpha}\right]\sin\frac{1}{\alpha} + \frac{\alpha^2}{2}\left[-2\sin\frac{2k}{\alpha}\sin\frac{1}{\alpha} + \ln\tan\frac{2k+1}{2\alpha} - \ln\tan\frac{2k-1}{2\alpha}\right]\right\}, \tag{182}$$

$$C^{ww}_{k\,k+2} = \frac{1}{2}\left(\frac{h}{a}\right)^2\left[\frac{\alpha^4}{4}\sin\frac{2(k+1)}{\alpha} + \frac{\nu^2\alpha^3}{2}\cos\frac{2(k+1)}{\alpha}\right]\sin\frac{1}{\alpha}, \tag{183}$$

and for N-2 and N-1,

$$C^{ww}_{N-2\,N-2} = 8(1+\nu)\sin\frac{2(N-2)}{\alpha}\sin\frac{1}{\alpha} + \frac{1}{12}\left(\frac{h}{a}\right)^2\left\{\frac{\alpha^4}{4}\left[\sin\frac{2(N-3)}{\alpha} + 4\sin\frac{2(N-2)}{\alpha} + \sin\frac{2(N-1)}{\alpha}\right]\sin\frac{1}{\alpha} + \nu^2\alpha^3\left[\cos\frac{2(N-3)}{\alpha} + 2\cos\frac{2(N-2)}{\alpha}\right]\sin\frac{1}{\alpha} + \frac{\alpha^2}{2}\left[-2\left(\sin\frac{2(N-3)}{\alpha} + \sin\frac{2(N-2)}{\alpha}\right)\sin\frac{1}{\alpha} + \ln\tan\frac{2N-3}{2\alpha} - \ln\tan\frac{2N-7}{2\alpha}\right]\right\}, \tag{184}$$

$$C^{ww}_{N-2\,N-1} = -\frac{1}{12}\left(\frac{h}{a}\right)^2\left\{\frac{\alpha^4}{2}\left[\sin\frac{2(N-2)}{\alpha} + \sin\frac{2(N-1)}{\alpha}\right]\sin\frac{1}{\alpha} + \frac{\nu^2\alpha^3}{2}\left[3\cos\frac{2(N-2)}{\alpha} + \cos\frac{2(N-1)}{\alpha}\right]\sin\frac{1}{\alpha} + \frac{\alpha^2}{2}\left[-2\sin\frac{2(N-2)}{\alpha}\sin\frac{1}{\alpha} + \ln\tan\frac{2N-3}{2\alpha} - \ln\tan\frac{2N-5}{2\alpha}\right]\right\}, \tag{185}$$

$$
\begin{aligned}
C^{ww}_{N-1\,N-1} &= 8(1+\nu)\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha \\
&+\frac{1}{12}\left(\frac{h}{a}\right)^2\Bigg\{\frac{\alpha^4}{4}\left[\sin\frac{2(N-2)}{\alpha}+4\sin\frac{2(N-1)}{\alpha}\right]\sin 1/\alpha \\
&+\frac{\alpha^2}{8}\left[(\alpha-2\nu\cot\bar{\theta})^2-2\alpha(\alpha-2\nu\cot\bar{\theta})\right]\left[\sin\bar{\theta}\sin 1/\alpha\right. \\
&\left.-(1-\cos 1/\alpha)\cos\bar{\theta}\right]+\nu^2\alpha^3\left[\cos\frac{2(N-2)}{\alpha}+2\cos\frac{2(N-1)}{\alpha}\right]\sin 1/\alpha \\
&-\nu^3\alpha^2\cot\bar{\theta}\left[\cos\bar{\theta}\sin 1/\alpha+(1-\cos 1/\alpha)\sin\bar{\theta}\right] \\
&+\frac{\alpha^2}{\alpha}\left[-2\sin\frac{2(N-2)}{\alpha}\sin 1/\alpha+\ln\tan\frac{2N-3}{2\alpha}-\ln\tan\frac{2N-5}{2\alpha}\right] \\
&+\frac{\alpha^2}{2}(1.000)\left[-2\sin\frac{2(N-1)}{\alpha}\sin 1/\alpha+\ln\tan\frac{2N-1}{2\alpha}\right. \\
&\left.-\ln\tan\frac{2N-3}{2\alpha}\right]+\frac{\alpha^2}{2}\left[(1-\cos 1/\alpha)\cos\bar{\theta}\right. \\
&\left.-\sin\bar{\theta}\sin 1/\alpha+\ln\tan\frac{\bar{\theta}}{2}-\ln\tan\frac{2N-1}{2\alpha}\right]\Bigg\}\,.
\end{aligned}
\tag{186}
$$

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